Computer Algebra
Independent Integration Tests

Summer 2023 edition

4-Trig-functions/4.2-Cosine/88-4.2.1.3-g-tan-^p-a+b-cos-^m

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Contents

CHAPTER 1

INTRODUCTION

This report gives the result of running the computer algebra independent integration test.The download section in on the main webpage contains links to download the problems in plain text format used for all CAS systems. The number of integrals in this report is [22]. This is test number [88].

1.1 Listing of CAS systems tested

The following are the CAS systems tested:

1. Mathematica 13.3.1 (August 16, 2023) on windows 10.
2. Rubi 4.16.1 (Dec 19, 2018) on Mathematica 13.3 on windows 10
3. Maple 2023.1 (July, 12, 2023) on windows 10.
4. Maxima 5.47 (June 1, 2023) using Lisp SBCL 2.3.0 on Linux via sagemath 10.1 (Aug 20, 2023).
5. FriCAS 1.3.9 (July 8, 2023) based on sbcl 2.3.0 on Linux via sagemath 10.1 (Aug 20, 2023).
6. Giac/Xcas 1.9.0-57 (June 26, 2023) on Linux via sagemath 10.1 (Aug 20, 2023).
7. Sympy 1.12 (May 10, 2023) Using Python 3.11.3 on Linux.
8. Mupad using Matlab 2021a with Symbolic Math Toolbox Version 8.7 on windows 10.

Maxima and Fricas and Giac are called using Sagemath. This was done using Sagemath `integrate` command by changing the name of the algorithm to use the different CAS systems.

Sympy was run directly in Python not via sagemath.

1.2 Results

Important note: A number of problems in this test suite have no antiderivative in closed form. This means the antiderivative of these integrals can not be expressed in terms of elementary, special functions or `Hypergeometric2F1` functions. `RootSum` and `RootOf` are not allowed. If a CAS returns the above integral unevaluated within the time limit, then the result is counted as passed and assigned an A grade.

However, if CAS times out, then it is assigned an F grade even if the integral is not integrable, as this implies CAS could not determine that the integral is not integrable in the time limit.

If a CAS returns an antiderivative to such an integral, it is assigned an A grade automatically and this special result is listed in the introduction section of each individual test report to make it easy to identify as this can be important result to investigate.

The results given in in the table below reflects the above.

System	% solved	% Failed
Rubi	100.00 (22)	0.00 (0)
Mathematica	100.00 (22)	0.00 (0)
Maple	100.00 (22)	0.00 (0)
Fricas	95.45 (21)	4.55 (1)
Giac	95.45 (21)	4.55 (1)
Mupad	81.82 (18)	18.18 (4)
Maxima	77.27 (17)	22.73 (5)
Sympy	4.55 (1)	95.45 (21)

Table 1.1: Percentage solved for each CAS

The table below gives additional break down of the grading of quality of the antiderivatives generated by each CAS. The grading is given using the letters A,B,C and F with A being the best quality. The grading is accomplished by comparing the antiderivative generated with the optimal antiderivatives included in the test suite. The following table describes the meaning of these grades.

grade	description
A	Integral was solved and antiderivative is optimal in quality and leaf size.
B	Integral was solved and antiderivative is optimal in quality but leaf size is larger than twice the optimal antiderivatives leaf size.
C	Integral was solved and antiderivative is non-optimal in quality. This can be due to one or more of the following reasons 1. antiderivative contains a hypergeometric function and the optimal antiderivative does not. 2. antiderivative contains a special function and the optimal antiderivative does not. 3. antiderivative contains the imaginary unit and the optimal antiderivative does not.
F	Integral was not solved. Either the integral was returned unevaluated within the time limit, or it timed out, or CAS hanged or crashed or an exception was raised.

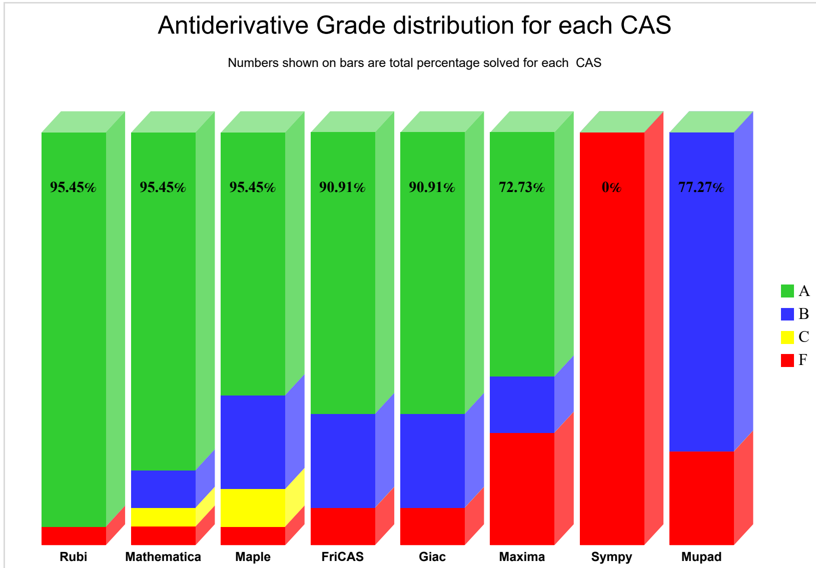
Table 1.2: Description of grading applied to integration result

Grading is implemented for all CAS systems. Based on the above, the following table summarizes the grading for this test suite.

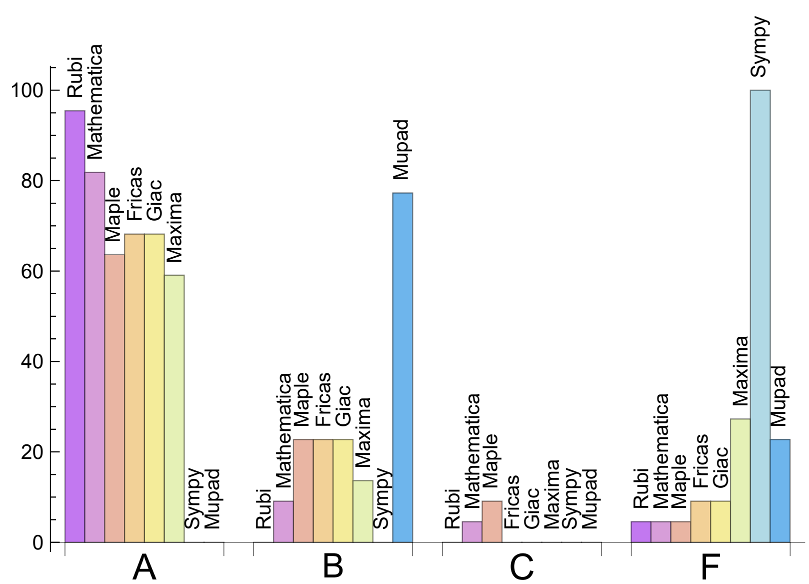
System	% A grade	% B grade	% C grade	% F grade
Rubi	95.455	0.000	0.000	4.545
Mathematica	81.818	9.091	4.545	4.545
Fricas	68.182	22.727	0.000	9.091
Giac	68.182	22.727	0.000	9.091
Maple	63.636	22.727	9.091	4.545
Maxima	59.091	13.636	0.000	27.273
Mupad	0.000	77.273	0.000	22.727
Sympy	0.000	0.000	0.000	100.000

Table 1.3: Antiderivative Grade distribution of each CAS

The following is a Bar chart illustration of the data in the above table.



The figure below compares the grades of the CAS systems.



The following table shows the distribution of the different types of failures for each CAS. There are 3 types failures. The first is when CAS returns the input within the time limit, which means it could not solve it. This is the typical failure and given as **F**.

The second failure is due to time out. CAS could not solve the integral within the 3 minutes time limit which is assigned. This is assigned **F(-1)**.

The third is due to an exception generated, indicated as **F(-2)**. This most likely indicates an interface problem between sagemath and the CAS (applicable only to FriCAS, Maxima and Giac) or it could be an indication of an internal error in the CAS itself. This type of error requires more investigation to determine the cause.

System	Number failed	Percentage normal failure	Percentage timeout failure	Percentage exception failure
Rubi	0	0.00	0.00	0.00
Mathematica	0	0.00	0.00	0.00
Maple	0	0.00	0.00	0.00
Fricas	1	0.00	100.00	0.00
Giac	1	100.00	0.00	0.00
Mupad	4	0.00	100.00	0.00
Maxima	5	20.00	0.00	80.00
Sympy	21	100.00	0.00	0.00

Table 1.4: Failure statistics for each CAS

1.3 Time and leaf size Performance

The table below summarizes the performance of each CAS system in terms of time used and leaf size of results.

Mean size is the average leaf size produced by the CAS (before any normalization). The Normalized mean is relative to the mean size of the optimal anti-derivative given in the input files.

For example, if CAS has **Normalized mean** of 3, then the mean size of its leaf size is 3 times as large as the mean size of the optimal leaf size.

Median size is value of leaf size where half the values are larger than this and half are smaller (before any normalization). i.e. The Middle value.

Similarly the **Normalized median** is relative to the median leaf size of the optimal.

For example, if a CAS has Normalized median of 1.2, then its median is 1.2 as large as the median leaf size of the optimal.

System	Mean time (sec)
Rubi	0.16
Fricas	0.30
Maxima	0.32
Giac	0.37
Maple	1.05
Mathematica	1.13
Mupad	14.11
Sympy	73.31

Table 1.5: Time performance for each CAS

System	Mean size	Normalized mean	Median size	Normalized median
Sympy	22.00	0.96	22.00	0.96
Maxima	49.65	1.46	46.00	1.22
Rubi	55.09	1.00	38.50	1.00
Giac	67.57	1.34	50.00	1.16
Mathematica	70.59	1.26	45.00	1.01
Maple	84.41	1.54	56.00	1.02
Fricas	105.43	1.89	53.00	1.52
Mupad	145.94	1.96	45.50	1.15

Table 1.6: Leaf size performance for each CAS

1.4 Performance based on number of rules Rubi used

This section shows how each CAS performed based on the number of rules Rubi needed to solve the same integral. One diagram is given for each CAS.

On the y axis is the percentage solved which Rubi itself needed the number of rules given the x axis. These plots show that as more rules are needed then most CAS system percentage of solving decreases which indicates the integral is becoming more complicated to solve.

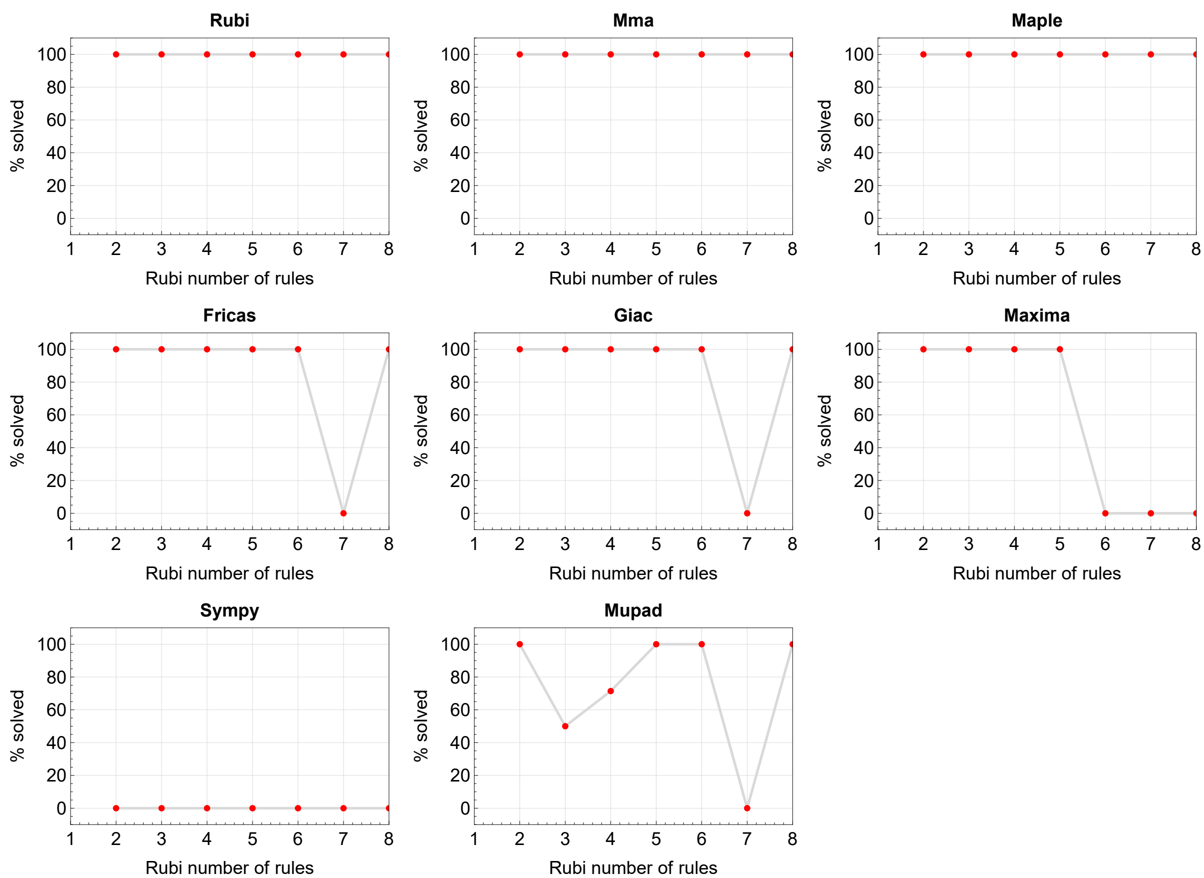


Figure 1.1: Solving statistics per number of Rubi rules used

1.5 Performance based on number of steps Rubi used

This section shows how each CAS performed based on the number of steps Rubi needed to solve the same integral. Note that the number of steps Rubi needed can be much higher than the number of rules, as the same rule could be used more than once.

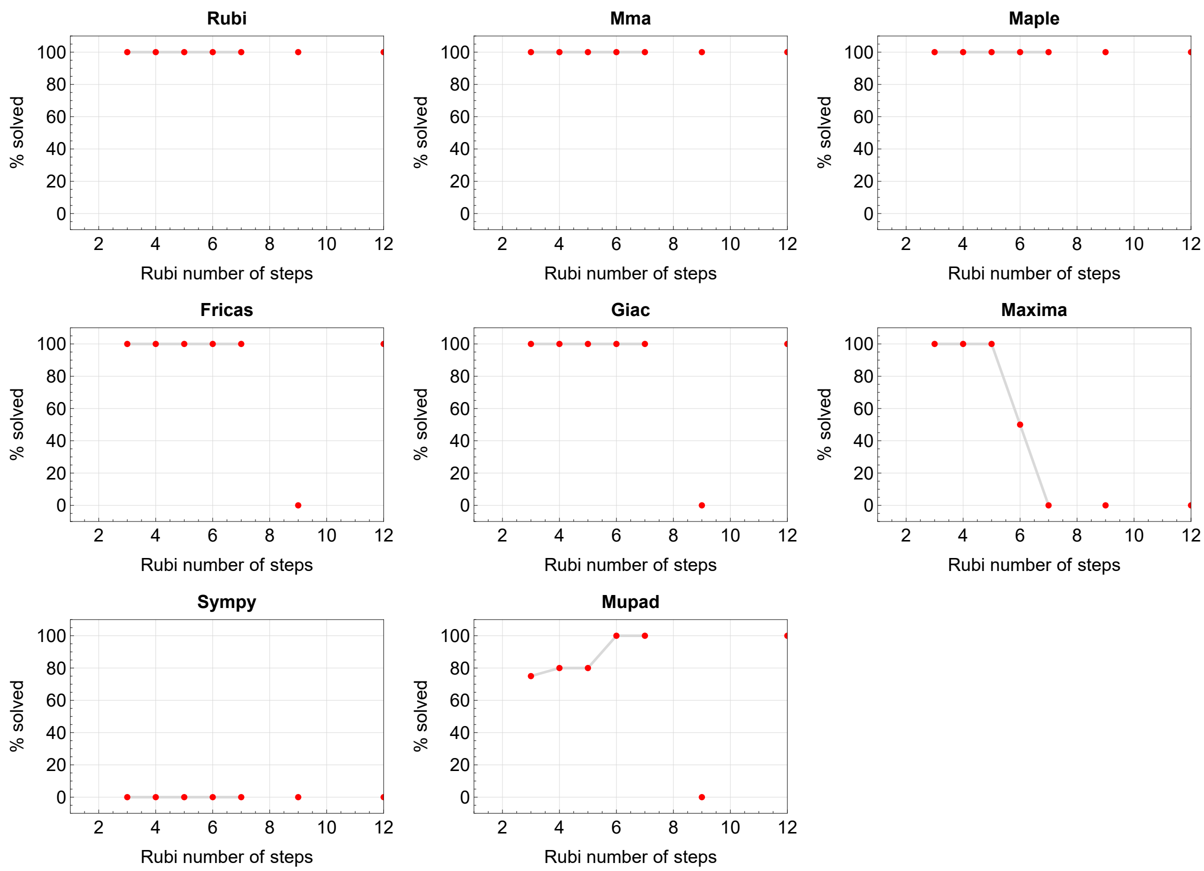


Figure 1.2: Solving statistics per number of Rubi steps used

The above diagram show that the precentage of solved intergals decreases for most CAS systems as the number of steps increases. As expected, for integrals that required less steps by Rubi, CAS systems had more success which indicates the integral was not as hard to solve. As Rubi needed more steps to solve the integral, the solved percentage decreased for most CAS systems which indicates the integral is becoming harder to solve.

1.6 Solved integrals histogram based on leaf size of result

The following shows the distribution of solved integrals for each CAS system based on leaf size of the antiderivatives produced by each CAS. It shows that most integrals solved produced leaf size less than about 100 to 150. The bin size used is 40.

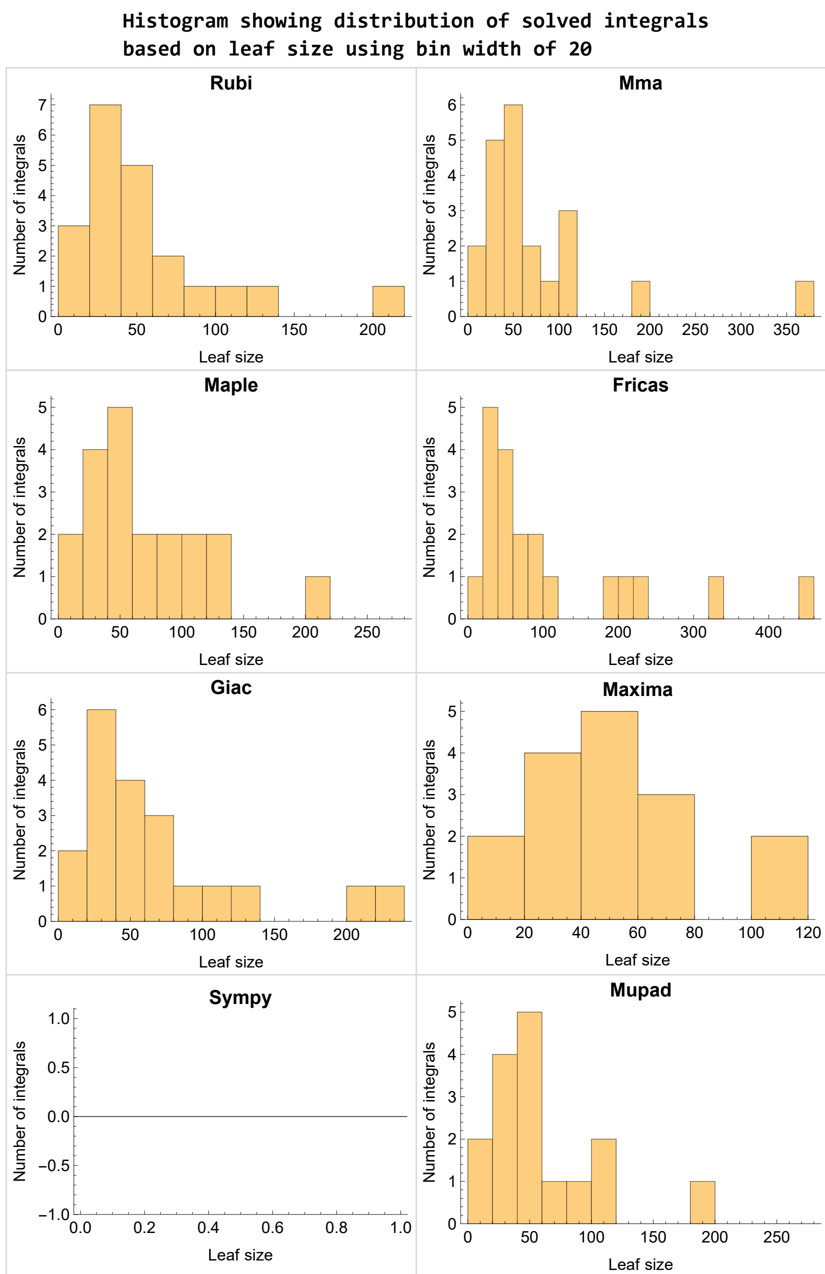


Figure 1.3: Solved integrals based on leaf size distribution

1.7 Solved integrals histogram based on CPU time used

The following shows the distribution of solved integrals for each CAS system based on CPU time used in seconds. The bin size used is 0.1 second.

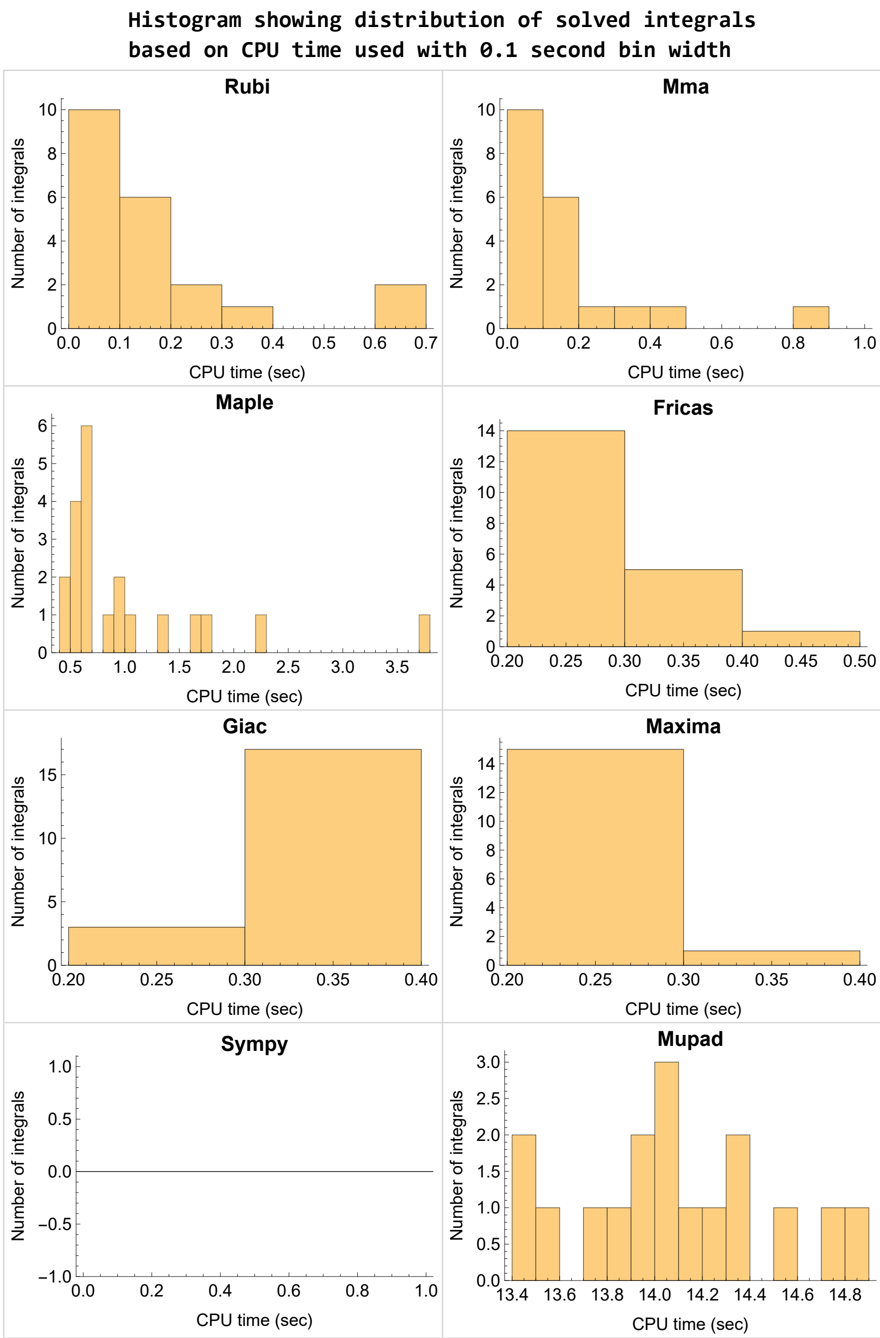


Figure 1.4: Solved integrals histogram based on CPU time used

1.8 Leaf size vs. CPU time used

The following shows the relation between the CPU time used to solve an integral and the leaf size of the antiderivative.

The result for Fricas, Maxima and Giac is shifted more to the right than the other CAS system due to the use of sagemath to call them, which causes an initial slight delay in the timing to start the integration due to overhead of starting a new process each time. This should also be taken into account when looking at the timing of these three CAS systems. Direct calls not using sagemath would result in faster timings, but current implementation uses sagemath as this makes testing much easier to do.

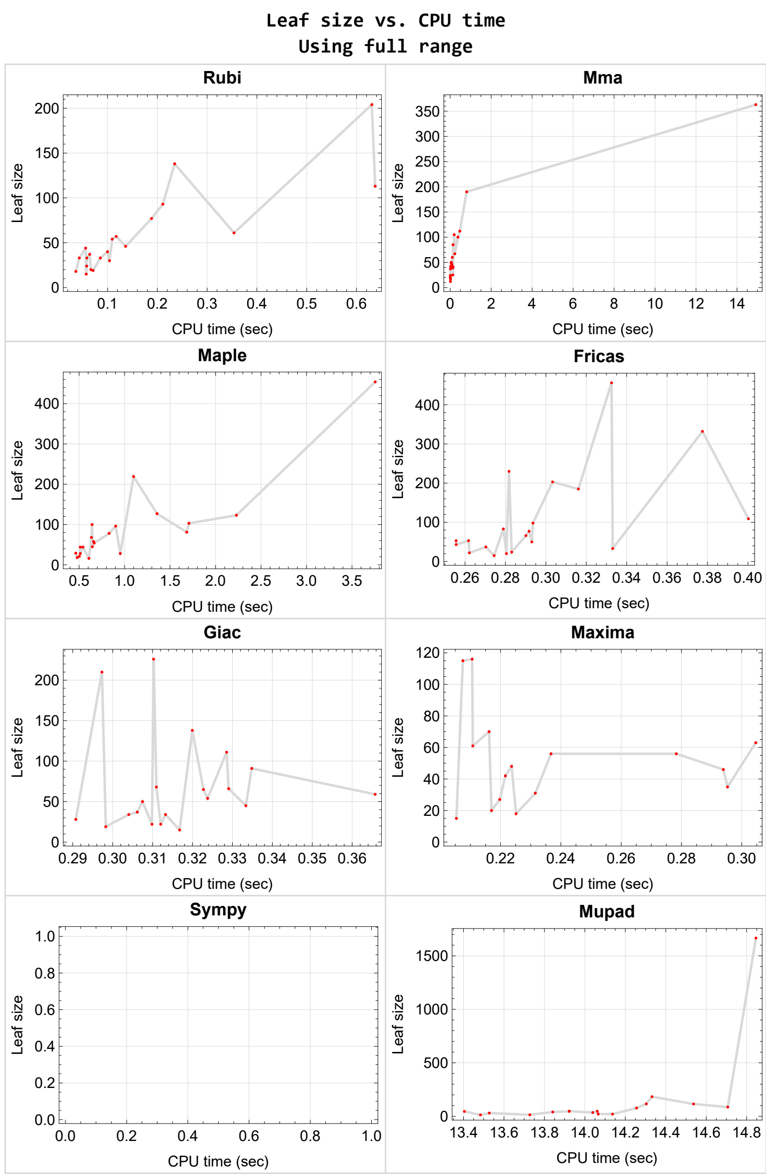


Figure 1.5: Leaf size vs. CPU time. Full range

1.9 list of integrals with no known antiderivative

{22}

1.10 List of integrals solved by CAS but has no known antiderivative

Rubi {}

Mathematica {}

Maple {}

Maxima {}

Fricas {}

Sympy {}

Giac {}

Mupad {}

1.11 list of integrals solved by CAS but failed verification

The following are integrals solved by CAS but the verification phase failed to verify the anti-derivative produced is correct. This does not necessarily mean that the anti-derivative is wrong as additional methods of verification might be needed, or more time is needed (3 minutes time limit was used). These integrals are listed here to make it possible to do further investigation to determine why the result could not be verified.

Rubi {}

Mathematica {21}

Maple {21}

Maxima Verification phase not currently implemented.

Fricas Verification phase not currently implemented.

Sympy Verification phase not currently implemented.

Giac Verification phase not currently implemented.

Mupad Verification phase not currently implemented.

1.12 Timing

The command `AbsoluteTiming[]` was used in Mathematica to obtain the elapsed time for each integrate call. In Maple, the command `Usage` was used as in the following example

```
cpu_time := Usage(assign ('result_of_int',int(expr,x)),output='realtime'
```

For all other CAS systems, the elapsed time to complete each integral was found by taking the difference between the time after the call completed from the time before the call was made. This was done using Python's `time.time()` call.

All elapsed times shown are in seconds. A time limit of 3 CPU minutes was used for each integral. If the integrate command did not complete within this time limit, the integral was aborted and considered to have failed and assigned an F grade. The time used by failed integrals due to time out was not counted in the final statistics.

1.13 Verification

A verification phase was applied on the result of integration for `Rubi` and `Mathematica`.

Future version of this report will implement verification for the other CAS systems. For the integrals whose result was not run through a verification phase, it is assumed that the antiderivative was correct.

Verification phase also had 3 minutes time out. An integral whose result was not verified could still be correct, but further investigation is needed on those integrals. These integrals were marked in the summary table below and also in each integral separate section so they are easy to identify and locate.

1.14 Important notes about some of the results

Important note about Maxima results

Since tests were run in a batch mode, and using an automated script, then any integral where Maxima needed an interactive response from the user to answer a question during the evaluation of the integral will fail.

The exception raised is `ValueError`. Therefore Maxima results is lower than what would result if Maxima was run directly and each question was answered correctly.

The percentage of such failures were not counted for each test file, but for an example, for the `Timofeev` test file, there were about 14 such integrals out of total 705, or about 2 percent. This percentage can be higher or lower depending on the specific input test file.

Such integrals can be identified by looking at the output of the integration in each section for Maxima. The exception message will indicate the cause of error.

Maxima integrate was run using SageMath with the following settings set by default

```
'besselexpand : true'
'display2d : false'
'domain : complex'
'keepfloat : true'
'load(to_poly_solve)'
'load(simplify_sum)'
'load(abs_integrate)' 'load(diag)'
```

SageMath automatic loading of Maxima `abs_integrate` was found to cause some problems. So the following code was added to disable this effect.

```
from sage.interfaces.maxima_lib import maxima_lib
maxima_lib.set('extra_definite_integration_methods', '[]')
maxima_lib.set('extra_integration_methods', '[]')
```

See `https://ask.sagemath.org/question/43088/integrate-results-that-are-different-from-using-maxima/` for reference.

Important note about FriCAS result

There were few integrals which failed due to SageMath interface and not because FriCAS system could not do the integration.

These will fail With error `Exception raised: NotImplementedError`.

The number of such cases seems to be very small. About 1 or 2 percent of all integrals. These can be identified by looking at the exception message given in the result.

Important note about finding leaf size of antiderivative

For Mathematica, Rubi, and Maple, the builtin system function `LeafSize` was used to find the leaf size of each antiderivative.

The other CAS systems (SageMath and Sympy) do not have special builtin function for this purpose at this time. Therefore the leaf size for Fricas and Sympy antiderivative was determined using the following function, thanks to user `slelievre` at `https://ask.sagemath.org/question/57123/could-we-have-a-leaf_count-function-in-base-sagemath/`

```
def tree_size(expr):
    r"""
    Return the tree size of this expression.
    """
    if expr not in SR:
        # deal with lists, tuples, vectors
        return 1 + sum(tree_size(a) for a in expr)
    expr = SR(expr)
```

```
  x, aa = expr.operator(), expr.operands()
  if x is None:
     return 1
  else:
     return 1 + sum(tree_size(a) for a in aa)
```

For Sympy, which was called directly from Python, the following code was used to obtain the leafsize of its result

```
try:
  # 1.7 is a fudge factor since it is low side from actual leaf count
  leafCount = round(1.7*count_ops(anti))

  except Exception as ee:
       leafCount =1
```

Important note about Mupad results

Matlab's symbolic toolbox does not have a leaf count function to measure the size of the antiderivative. Maple was used to determine the leaf size of Mupad output by post processing Mupad result.

Currently no grading of the antiderivative for Mupad is implemented. If it can integrate the problem, it was assigned a B grade automatically as a placeholder. In the future, when grading function is implemented for Mupad, the tests will be rerun again.

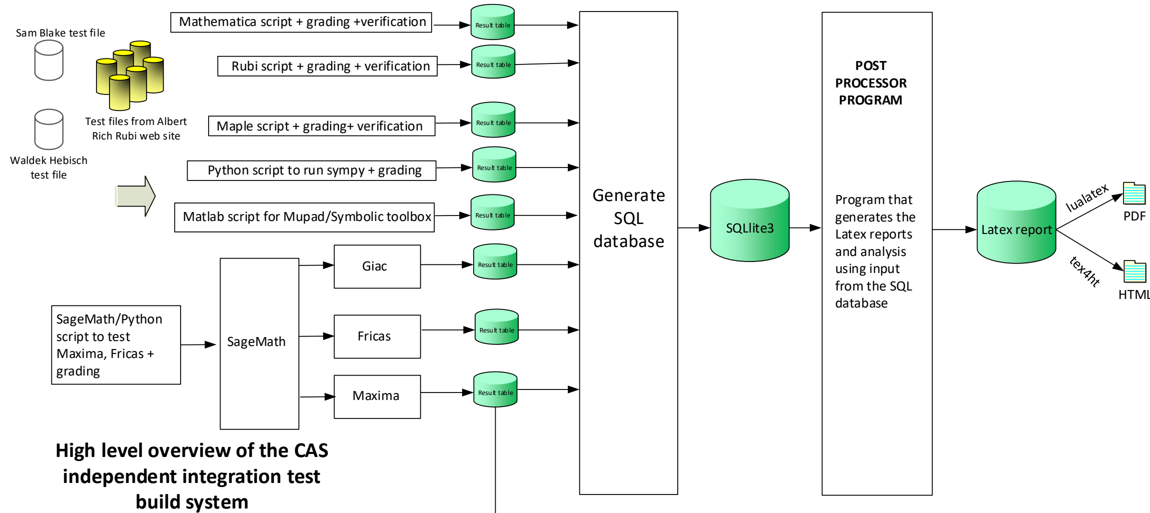
The following is an example of using Matlab's symbolic toolbox (Mupad) to solve an integral

```
integrand = evalin(symengine,'cos(x)*sin(x)')
the_variable = evalin(symengine,'x')
anti = int(integrand,the_variable)
```

Which gives `sin(x)^2/2`

1.15 Design of the test system

The following diagram gives a high level view of the current test build system.



One record (line) per one integral result. The line is CSV comma separated. This is description of each record

1. integer, the problem number.
2. integer. 0 for failed, 1 for passed, -1 for timeout, -2 for CAS specific exception. (this is not the grade field)
3. integer. Leaf size of result.
4. integer. Leaf size of the optimal antiderivative.
5. number. CPU time used to solve this integral. 0 if failed.
6. string. The integral in Latex format
7. string. The input used in CAS own syntax.
8. string. The result (antiderivative) produced by CAS in Latex format
9. string. The optimal antiderivative in Latex format.
10. integer. 0 or 1. Indicates if problem has known antiderivative or not
11. String. The result (antiderivative) in CAS own syntax.
12. String. The grade of the antiderivative. Can be “A”, “B”, “C”, or “F”
13. String. Small string description of why the grade was given.
14. integer. 1 if result was verified or 0 if not verified. (For mma, rubi and maple only)

*The following fields are present only in **Rubi** Table file*

15. integer. Number of steps used.
16. integer. Number of rules used.
17. integer. Integrand leaf size.
18. real number. Ratio. Field 16 over field 17
19. String of form “{n,n,..}” which is list of the rules used by Rubi
20. String. The optimal antiderivative in Mathematica syntx

Nasser M. Abbasi
June 27, 2023
Design.vsdx

CHAPTER 2

DETAILED SUMMARY TABLES OF RESULTS

2.1 List of integrals sorted by grade for each CAS

Rubi

A grade { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21 }

B grade { }

C grade { }

F normal fail { }

F(-1) timedout fail { }

F(-2) exception fail { }

Mma

A grade { 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20 }

B grade { 1, 3 }

C grade { 21 }

F normal fail { }

F(-1) timedout fail { }

F(-2) exception fail { }

Maple

A grade $\{ 2, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17 \}$

B grade $\{ 10, 18, 19, 20, 21 \}$

C grade $\{ 1, 3 \}$

F normal fail $\{ \}$

F(-1) timedout fail $\{ \}$

F(-2) exception fail $\{ \}$

Fricas

A grade $\{ 1, 2, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 17, 19 \}$

B grade $\{ 3, 7, 16, 18, 20 \}$

C grade $\{ \}$

F normal fail $\{ \}$

F(-1) timedout fail $\{ 21 \}$

F(-2) exception fail $\{ \}$

Maxima

A grade $\{ 2, 4, 5, 6, 7, 9, 11, 13, 14, 16, 18, 19, 20 \}$

B grade $\{ 1, 3, 8 \}$

C grade $\{ \}$

F normal fail $\{ 21 \}$

F(-1) timedout fail $\{ \}$

F(-2) exception fail $\{ 10, 12, 15, 17 \}$

Giac

A grade $\{ 2, 4, 5, 6, 7, 8, 9, 11, 13, 14, 15, 16, 17, 19, 20 \}$

B grade $\{ 1, 3, 10, 12, 18 \}$

C grade $\{ \}$

F normal fail $\{ 21 \}$

F(-1) timedout fail $\{ \}$

F(-2) exception fail $\{ \}$

Mupad

A grade $\{\ \}$

B grade $\{\ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17\ \}$

C grade $\{\ \}$

F normal fail $\{\ \}$

F(-1) timedout fail $\{\ 18, 19, 20, 21\ \}$

F(-2) exception fail $\{\ \}$

Sympy

A grade $\{\ \}$

B grade $\{\ \}$

C grade $\{\ \}$

F normal fail $\{\ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\ \}$

F(-1) timedout fail $\{\ \}$

F(-2) exception fail $\{\ \}$

2.2 Detailed conclusion table per each integral for all CAS systems

Detailed conclusion table per each integral is given by the table below. The elapsed time is in seconds. For failed result it is given as **F(-1)** if the failure was due to timeout. It is given as **F(-2)** if the failure was due to an exception being raised, which could indicate a bug in the system. If the failure was due to integral not being evaluated within the time limit, then it is given as **F**.

In this table, the column `N.S.` means **normalized size** and is defined as $\frac{\text{antiderivative leaf size}}{\text{optimal antiderivative leaf size}}$. To make the table fit the page, the name **`Mathematica`** was abbreviated to **`MMA`**.

Problem 1	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	B	C	B	A	F	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	33	33	105	68	115	50	0	65	46
N.S.	1	1.00	3.18	2.06	3.48	1.52	0.00	1.97	1.39
time (sec)	N/A	0.086	0.198	0.636	0.207	0.293	0.000	0.323	13.922

Problem 2	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	19	19	17	18	15	15	0	15	13
N.S.	1	1.00	0.89	0.95	0.79	0.79	0.00	0.79	0.68
time (sec)	N/A	0.071	0.018	0.479	0.205	0.274	0.000	0.317	13.484

Problem 3	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	B	C	B	B	F	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	15	15	39	44	61	33	0	45	30
N.S.	1	1.00	2.60	2.93	4.07	2.20	0.00	3.00	2.00
time (sec)	N/A	0.057	0.132	0.513	0.211	0.333	0.000	0.333	13.527

Problem 4	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	18	18	12	16	18	20	0	19	14
N.S.	1	1.00	0.67	0.89	1.00	1.11	0.00	1.06	0.78
time (sec)	N/A	0.036	0.013	0.608	0.225	0.280	0.000	0.298	13.728

Problem 5	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	33	33	42	28	31	37	0	34	21
N.S.	1	1.00	1.27	0.85	0.94	1.12	0.00	1.03	0.64
time (sec)	N/A	0.059	0.027	0.514	0.232	0.270	0.000	0.304	14.066

Problem 6	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	30	30	25	29	42	24	0	37	35
N.S.	1	1.00	0.83	0.97	1.40	0.80	0.00	1.23	1.17
time (sec)	N/A	0.104	0.130	0.464	0.222	0.283	0.000	0.306	14.039

Problem 7	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	B	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	46	46	60	44	56	83	0	50	40
N.S.	1	1.00	1.30	0.96	1.22	1.80	0.00	1.09	0.87
time (sec)	N/A	0.136	0.104	0.543	0.237	0.279	0.000	0.307	13.841

Problem 8	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	B	A	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	40	40	41	45	70	53	0	59	45
N.S.	1	1.00	1.02	1.12	1.75	1.32	0.00	1.48	1.12
time (sec)	N/A	0.100	0.142	0.646	0.216	0.256	0.000	0.366	13.404

Problem 9	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	33	33	49	28	27	43	0	28	20
N.S.	1	1.00	1.48	0.85	0.82	1.30	0.00	0.85	0.61
time (sec)	N/A	0.043	0.045	0.953	0.220	0.256	0.000	0.291	14.137

Problem 10	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	B	F(-2)	A	F	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	113	113	190	219	0	332	0	226	1666
N.S.	1	1.00	1.68	1.94	0.00	2.94	0.00	2.00	14.74
time (sec)	N/A	0.637	0.806	1.099	0.000	0.377	0.000	0.310	14.846

Problem 11	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	57	57	46	58	56	66	0	66	115
N.S.	1	1.00	0.81	1.02	0.98	1.16	0.00	1.16	2.02
time (sec)	N/A	0.117	0.083	0.659	0.278	0.290	0.000	0.329	14.537

Problem 12	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	F(-2)	A	F	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	61	61	85	100	0	203	0	111	77
N.S.	1	1.00	1.39	1.64	0.00	3.33	0.00	1.82	1.26
time (sec)	N/A	0.354	0.142	0.643	0.000	0.303	0.000	0.329	14.256

Problem 13	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	20	20	20	21	20	22	0	22	48
N.S.	1	1.00	1.00	1.05	1.00	1.10	0.00	1.10	2.40
time (sec)	N/A	0.067	0.007	0.503	0.217	0.262	0.000	0.310	13.923

Problem 14	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	54	54	50	54	48	53	0	54	47
N.S.	1	1.00	0.93	1.00	0.89	0.98	0.00	1.00	0.87
time (sec)	N/A	0.109	0.059	0.666	0.224	0.262	0.000	0.324	14.061

Problem 15	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	F(-2)	A	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	77	77	67	78	0	230	0	91	86
N.S.	1	1.00	0.87	1.01	0.00	2.99	0.00	1.18	1.12
time (sec)	N/A	0.188	0.227	0.830	0.000	0.282	0.000	0.335	14.707

Problem 16	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	B	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	93	93	100	96	116	185	0	138	116
N.S.	1	1.00	1.08	1.03	1.25	1.99	0.00	1.48	1.25
time (sec)	N/A	0.211	0.380	0.902	0.211	0.316	0.000	0.320	14.304

Problem 17	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	F(-2)	A	F	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	138	138	112	127	0	456	0	210	183
N.S.	1	1.00	0.81	0.92	0.00	3.30	0.00	1.52	1.33
time (sec)	N/A	0.235	0.470	1.357	0.000	0.332	0.000	0.297	14.332

Problem 18	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	B	A	B	F	B	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	44	44	44	81	63	77	0	68	0
N.S.	1	1.00	1.00	1.84	1.43	1.75	0.00	1.55	0.00
time (sec)	N/A	0.056	0.083	1.683	0.305	0.292	0.000	0.311	0.000

Problem 19	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	B	A	A	F	A	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	37	37	37	123	46	109	0	34	0
N.S.	1	1.00	1.00	3.32	1.24	2.95	0.00	0.92	0.00
time (sec)	N/A	0.064	0.014	2.230	0.294	0.400	0.000	0.313	0.000

Problem 20	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	B	A	B	F	A	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	24	24	24	103	35	98	0	22	0
N.S.	1	1.00	1.00	4.29	1.46	4.08	0.00	0.92	0.00
time (sec)	N/A	0.058	0.008	1.707	0.295	0.294	0.000	0.312	0.000

Problem 21	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	B	F	F(-1)	F	F	F(-1)
verified	N/A	Yes	No	No	TBD	TBD	TBD	TBD	TBD
size	204	204	363	454	0	0	0	0	0
N.S.	1	1.00	1.78	2.23	0.00	0.00	0.00	0.00	0.00
time (sec)	N/A	0.631	14.916	3.755	0.000	0.000	0.000	0.000	0.000

Problem 22	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
verified	N/A	N/A	N/A	N/A	TBD	TBD	TBD	TBD	TBD
size	23	23	25	23	25	25	22	25	25
N.S.	1	1.00	1.09	1.00	1.09	1.09	0.96	1.09	1.09
time (sec)	N/A	0.117	6.954	1.618	1.679	0.439	73.314	1.530	14.783

2.3 Detailed conclusion table specific for Rubi results

The following table is specific to Rubi only. It gives additional statistics for each integral. the column **steps** is the number of steps used by Rubi to obtain the antiderivative. The **rules** column is the number of unique rules used. The **integrand size** column is the leaf size of the integrand. Finally the ratio $\frac{\text{number of rules}}{\text{integrand size}}$ is also given. The larger this ratio is, the harder the integral is to solve. In this test file, problem number [17] had the largest ratio of [.6153999999999999947]

Table 2.1: Rubi specific breakdown of results for each integral

#	grade	number of steps used	number of unique rules	normalized antiderivative leaf size	integrand leaf size	$\frac{\text{number of rules}}{\text{integrand leaf size}}$
1	A	5	5	1.00	13	0.385
2	A	5	4	1.00	13	0.308
3	A	4	4	1.00	13	0.308
4	A	4	4	1.00	11	0.364
5	A	5	5	1.00	11	0.454
6	A	5	4	1.00	13	0.308
7	A	6	5	1.00	13	0.385
8	A	6	5	1.00	13	0.385
9	A	3	2	1.00	13	0.154
10	A	6	6	1.00	13	0.462
11	A	3	2	1.00	13	0.154
12	A	6	6	1.00	13	0.462
13	A	4	4	1.00	11	0.364
14	A	3	2	1.00	11	0.182
15	A	7	6	1.00	13	0.462
16	A	4	3	1.00	13	0.231
17	A	12	8	1.00	13	0.615
18	A	5	4	1.00	13	0.308
19	A	4	4	1.00	13	0.308
20	A	3	3	1.00	13	0.231
21	A	9	7	1.00	25	0.280
22	N/A	0	0	1.00	23	0.000

CHAPTER 3

LISTING OF INTEGRALS

3.1 $\int \frac{\tan^4(x)}{a+a\cos(x)}\,dx$

Optimal result

Integrand size = 13, antiderivative size = 33

$$\int \frac{\tan^4(x)}{a+a\cos(x)}\,dx=\frac{\operatorname{arctanh}(\sin(x))}{2a}-\frac{\sec(x)\tan(x)}{2a}+\frac{\tan^3(x)}{3a}$$

```
[Out] 1/2*arctanh(sin(x))/a-1/2*sec(x)*tan(x)/a+1/3*tan(x)^3/a
```

Rubi [A] (verified)

Time = 0.09 (sec) , antiderivative size = 33, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.385, Rules used = {2785, 2687, 30, 2691, 3855}

$$\int \frac{\tan^4(x)}{a+a\cos(x)}\,dx=\frac{\operatorname{arctanh}(\sin(x))}{2a}+\frac{\tan^3(x)}{3a}-\frac{\tan(x)\sec(x)}{2a}$$

```
[In] Int[Tan[x]^4/(a + a*Cos[x]),x]
```

```
[Out] ArcTanh[Sin[x]]/(2*a) - (Sec[x]*Tan[x])/(2*a) + Tan[x]^3/(3*a)
```

Rule 30

```
Int[(x_)^(m_.), x_Symbol] :> Simp[x^(m + 1)/(m + 1), x] /; FreeQ[m, x] && N
eQ[m, -1]
```

Rule 2687

```
Int[sec[(e_.) + (f_.)*(x_)]^(m_)*((b_.)*tan[(e_.) + (f_.)*(x_)])^(n_.), x_S
ymbol] :> Dist[1/f, Subst[Int[(b*x)^n*(1 + x^2)^(m/2 - 1), x], x, Tan[e + f
*x]], x] /; FreeQ[{b, e, f, n}, x] && IntegerQ[m/2] && !(IntegerQ[(n - 1)/
```

```
2] && LtQ[0, n, m - 1])
```

Rule 2691

```
Int[((a_.)*sec[(e_.) + (f_.)*(x_)])^(m_.)*((b_.)*tan[(e_.) + (f_.)*(x_)])^(
n_), x_Symbol] :> Simp[b*(a*Sec[e + f*x])^m*((b*Tan[e + f*x])^(n - 1)/(f*(m
 + n - 1))), x] - Dist[b^2*((n - 1)/(m + n - 1)), Int[(a*Sec[e + f*x])^m*(b
*Tan[e + f*x])^(n - 2), x], x] /; FreeQ[{a, b, e, f, m}, x] && GtQ[n, 1] &&
 NeQ[m + n - 1, 0] && IntegersQ[2*m, 2*n]
```

Rule 2785

```
Int[((g_.)*tan[(e_.) + (f_.)*(x_)])^(p_.)/((a_) + (b_.)*sin[(e_.) + (f_.)*(
x_)]), x_Symbol] :> Dist[1/a, Int[Sec[e + f*x]^2*(g*Tan[e + f*x])^p, x], x]
 - Dist[1/(b*g), Int[Sec[e + f*x]*(g*Tan[e + f*x])^(p + 1), x], x] /; FreeQ
[{a, b, e, f, g, p}, x] && EqQ[a^2 - b^2, 0] && NeQ[p, -1]
```

Rule 3855

```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] :> Simp[-ArcTanh[Cos[c + d*x]]/d, x]
 /; FreeQ[{c, d}, x]
```

Rubi steps

$$\begin{aligned}
\text{integral} &= -\frac{\int \sec(x)\tan^2(x)\,dx}{a}+\frac{\int \sec^2(x)\tan^2(x)\,dx}{a}\\
&= -\frac{\sec(x)\tan(x)}{2a}+\frac{\int \sec(x)\,dx}{2a}+\frac{\text{Subst}\left(\int x^2\,dx,x,\tan(x)\right)}{a}\\
&= \frac{\text{arctanh}(\sin(x))}{2a}-\frac{\sec(x)\tan(x)}{2a}+\frac{\tan^3(x)}{3a}
\end{aligned}$$

Mathematica [B] (verified)

Leaf count is larger than twice the leaf count of optimal. 105 vs. 2(33) = 66.

Time = 0.20 (sec) , antiderivative size = 105, normalized size of antiderivative = 3.18

$$\int \frac{\tan^4(x)}{a+a\cos(x)}\,dx = -\frac{\sec^3(x)\left(9\cos(x)\left(\log\left(\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right)-\log\left(\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right)\right)+3\cos(3x)\left(\log\left(\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right)\right.}{24a}$$

```
[In] Integrate[Tan[x]^4/(a + a*Cos[x]),x]
```

```
[Out] -1/24*(Sec[x]^3*(9*Cos[x]*(Log[Cos[x/2] - Sin[x/2]] - Log[Cos[x/2] + Sin[x/
2]]) + 3*Cos[3*x]*(Log[Cos[x/2] - Sin[x/2]] - Log[Cos[x/2] + Sin[x/2]]) + 2
*(-3*Sin[x] + 3*Sin[2*x] + Sin[3*x])))/a
```

Maple [C] (verified)

Result contains complex when optimal does not.

Time = 0.64 (sec) , antiderivative size = 68, normalized size of antiderivative = 2.06

method	result	[illegible]
risch	$\frac{i\left(3\,{\mathrm e}^{5 i x}-6\,{\mathrm e}^{4 i x}-3\,{\mathrm e}^{i x}-2\right)}{3\left({\mathrm e}^{2 i x}+1\right)^{3} a}+\frac{\ln\left({\mathrm e}^{i x}+i\right)}{2 a}-\frac{\ln\left({\mathrm e}^{i x}-i\right)}{2 a}$	[illegible]
default	$\frac{-\frac{1}{3\left(\tan\left(\frac{x}{2}\right)-1\right)^{3}}-\frac{1}{\left(\tan\left(\frac{x}{2}\right)-1\right)^{2}}-\frac{1}{2\left(\tan\left(\frac{x}{2}\right)-1\right)}-\frac{\ln\left(\tan\left(\frac{x}{2}\right)-1\right)}{2}-\frac{1}{3\left(\tan\left(\frac{x}{2}\right)+1\right)^{3}}+\frac{1}{\left(\tan\left(\frac{x}{2}\right)+1\right)^{2}}-\frac{1}{2\left(\tan\left(\frac{x}{2}\right)+1\right)}+\frac{\ln\left(\tan\left(\frac{x}{2}\right)+1\right)}{2}}{a}$	[illegible]

```
[In] int(tan(x)^4/(a+cos(x)*a),x,method=_RETURNVERBOSE)
```

```
[Out] 1/3*I*(3*exp(5*I*x)-6*exp(4*I*x)-3*exp(I*x)-2)/(exp(2*I*x)+1)^3/a+1/2/a*ln(
exp(I*x)+I)-1/2/a*ln(exp(I*x)-I)
```

Fricas [A] (verification not implemented)

none

Time = 0.29 (sec) , antiderivative size = 50, normalized size of antiderivative = 1.52

$$\int \frac{\tan^4(x)}{a+a\cos(x)}\,dx = \frac{3\,\cos\left(x\right)^{3}\log\left(\sin\left(x\right)+1\right)-3\,\cos\left(x\right)^{3}\log\left(-\sin\left(x\right)+1\right)-2\,\left(2\,\cos\left(x\right)^{2}+3\,\cos\left(x\right)-2\right)\sin\left(x\right)}{12\,a\cos\left(x\right)^{3}}$$

```
[In] integrate(tan(x)^4/(a+a*cos(x)),x, algorithm="fricas")
```

```
[Out] 1/12*(3*cos(x)^3*log(sin(x) + 1) - 3*cos(x)^3*log(-sin(x) + 1) - 2*(2*cos(x
)^2 + 3*cos(x) - 2)*sin(x))/(a*cos(x)^3)
```

Sympy [F]

$$\int \frac{\tan^4(x)}{a+a\cos(x)}\,dx = \frac{\int \frac{\tan^{4}{\left(x \right)}}{\cos{\left(x \right)}+1}\,dx}{a}$$

```
[In] integrate(tan(x)**4/(a+a*cos(x)),x)
```

```
[Out] Integral(tan(x)**4/(cos(x) + 1), x)/a
```

Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 115 vs. $2(27) = 54$.

Time = 0.21 (sec) , antiderivative size = 115, normalized size of antiderivative = 3.48

$$\int \frac{\tan^4(x)}{a + a\cos(x)}\,dx = -\frac{\frac{3\sin(x)}{\cos(x)+1} - \frac{8\sin(x)^3}{(\cos(x)+1)^3} - \frac{3\sin(x)^5}{(\cos(x)+1)^5}}{3\left(a - \frac{3a\sin(x)^2}{(\cos(x)+1)^2} + \frac{3a\sin(x)^4}{(\cos(x)+1)^4} - \frac{a\sin(x)^6}{(\cos(x)+1)^6}\right)} + \frac{\log\left(\frac{\sin(x)}{\cos(x)+1} + 1\right)}{2a} - \frac{\log\left(\frac{\sin(x)}{\cos(x)+1} - 1\right)}{2a}$$

```
[In] integrate(tan(x)^4/(a+a*cos(x)),x, algorithm="maxima")
[Out] -1/3*(3*sin(x)/(cos(x) + 1) - 8*sin(x)^3/(cos(x) + 1)^3 - 3*sin(x)^5/(cos(x
) + 1)^5)/(a - 3*a*sin(x)^2/(cos(x) + 1)^2 + 3*a*sin(x)^4/(cos(x) + 1)^4 -
a*sin(x)^6/(cos(x) + 1)^6) + 1/2*log(sin(x)/(cos(x) + 1) + 1)/a - 1/2*log(s
in(x)/(cos(x) + 1) - 1)/a
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 65 vs. $2(27) = 54$.

Time = 0.32 (sec) , antiderivative size = 65, normalized size of antiderivative = 1.97

$$\int \frac{\tan^4(x)}{a + a\cos(x)}\,dx = \frac{\log\left(\left|\tan\left(\frac{1}{2}x\right) + 1\right|\right)}{2a} - \frac{\log\left(\left|\tan\left(\frac{1}{2}x\right) - 1\right|\right)}{2a} - \frac{3\tan\left(\frac{1}{2}x\right)^5 + 8\tan\left(\frac{1}{2}x\right)^3 - 3\tan\left(\frac{1}{2}x\right)}{3\left(\tan\left(\frac{1}{2}x\right)^2 - 1\right)^3 a}$$

```
[In] integrate(tan(x)^4/(a+a*cos(x)),x, algorithm="giac")
[Out] 1/2*log(abs(tan(1/2*x) + 1))/a - 1/2*log(abs(tan(1/2*x) - 1))/a - 1/3*(3*ta
n(1/2*x)^5 + 8*tan(1/2*x)^3 - 3*tan(1/2*x))/((tan(1/2*x)^2 - 1)^3*a)
```

Mupad [B] (verification not implemented)

Time = 13.92 (sec) , antiderivative size = 46, normalized size of antiderivative = 1.39

$$\int \frac{\tan^4(x)}{a + a\cos(x)}\,dx = \frac{\operatorname{atanh}\left(\tan\left(\frac{x}{2}\right)\right)}{a} - \frac{\tan\left(\frac{x}{2}\right)^5 + \frac{8\tan\left(\frac{x}{2}\right)^3}{3} - \tan\left(\frac{x}{2}\right)}{a\left(\tan\left(\frac{x}{2}\right)^2 - 1\right)^3}$$

```
[In] int(tan(x)^4/(a + a*cos(x)),x)
[Out] atanh(tan(x/2))/a - ((8*tan(x/2)^3)/3 - tan(x/2) + tan(x/2)^5)/(a*(tan(x/2)
^2 - 1)^3)
```

3.2 $\int \frac{\tan^3(x)}{a+a\cos(x)}\,dx$

Optimal result

Integrand size = 13, antiderivative size = 19

$$\int \frac{\tan^3(x)}{a+a\cos(x)}\,dx = -\frac{\sec(x)}{a}+\frac{\sec^2(x)}{2a}$$

```
[Out] -sec(x)/a+1/2*sec(x)^2/a
```

Rubi [A] (verified)

Time = 0.07 (sec) , antiderivative size = 19, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 4, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.308, Rules used = {2785, 2686, 30, 8}

$$\int \frac{\tan^3(x)}{a+a\cos(x)}\,dx = \frac{\sec^2(x)}{2a}-\frac{\sec(x)}{a}$$

```
[In] Int[Tan[x]^3/(a + a*Cos[x]),x]
[Out] -(Sec[x]/a) + Sec[x]^2/(2*a)
```

Rule 8

```
Int[a_, x_Symbol] :> Simp[a*x, x] /; FreeQ[a, x]
```

Rule 30

```
Int[(x_)^(m_.), x_Symbol] :> Simp[x^(m + 1)/(m + 1), x] /; FreeQ[m, x] && N
eQ[m, -1]
```

Rule 2686

```
Int[((a_.)*sec[(e_.) + (f_.)*(x_)])^(m_.)*((b_.)*tan[(e_.) + (f_.)*(x_)])^(
n_.), x_Symbol] :> Dist[a/f, Subst[Int[(a*x)^(m - 1)*(-1 + x^2)^((n - 1)/2)
, x], x, Sec[e + f*x]], x] /; FreeQ[{a, e, f, m}, x] && IntegerQ[(n - 1)/2]
 &&  !(IntegerQ[m/2] && LtQ[0, m, n + 1])
```

Rule 2785

```
Int[((g_.)*tan[(e_.) + (f_.)*(x_)])^(p_.)/((a_) + (b_.)*sin[(e_.) + (f_.)*(
x_)]), x_Symbol] :> Dist[1/a, Int[Sec[e + f*x]^2*(g*Tan[e + f*x])^p, x], x]
 - Dist[1/(b*g), Int[Sec[e + f*x]*(g*Tan[e + f*x])^(p + 1), x], x] /; FreeQ
[{a, b, e, f, g, p}, x] && EqQ[a^2 - b^2, 0] && NeQ[p, -1]
```

Rubi steps

$$\begin{aligned}
\text{integral} &= -\frac{\int \sec(x)\tan(x)\,dx}{a} + \frac{\int \sec^2(x)\tan(x)\,dx}{a} \\
&= -\frac{\text{Subst}(\int 1\,dx, x, \sec(x))}{a} + \frac{\text{Subst}(\int x\,dx, x, \sec(x))}{a} \\
&= -\frac{\sec(x)}{a} + \frac{\sec^2(x)}{2a}
\end{aligned}$$

Mathematica [A] (verified)

Time = 0.02 (sec) , antiderivative size = 17, normalized size of antiderivative = 0.89

$$\int \frac{\tan^3(x)}{a + a\cos(x)}\,dx = \frac{2\sec^2(x)\sin^4\left(\frac{x}{2}\right)}{a}$$

```
[In] Integrate[Tan[x]^3/(a + a*Cos[x]),x]
[Out] (2*Sec[x]^2*Sin[x/2]^4)/a
```

Maple [A] (verified)

Time = 0.48 (sec) , antiderivative size = 18, normalized size of antiderivative = 0.95

method	result	size
default	$\frac{\frac{1}{2\cos(x)^2}-\frac{1}{\cos(x)}}{a}$	18
risch	$-\frac{2\left(e^{3ix}-e^{2ix}+e^{ix}\right)}{\left(e^{2ix}+1\right)^2 a}$	33

```
[In] int(tan(x)^3/(a+cos(x)*a),x,method=_RETURNVERBOSE)
```

```
[Out] 1/a*(1/2/cos(x)^2-1/cos(x))
```

Fricas [A] (verification not implemented)

none

Time = 0.27 (sec) , antiderivative size = 15, normalized size of antiderivative = 0.79

$$\int \frac{\tan^3(x)}{a+a\cos(x)}\,dx = -\frac{2\,\cos{(x)} - 1}{2\,a\cos{(x)}^2}$$

```
[In] integrate(tan(x)^3/(a+a*cos(x)),x, algorithm="fricas")
[Out] -1/2*(2*cos(x) - 1)/(a*cos(x)^2)
```

Sympy [F]

$$\int \frac{\tan^3(x)}{a+a\cos(x)}\,dx = \frac{\int \frac{\tan^3{(x)}}{\cos{(x)}+1}\,dx}{a}$$

```
[In] integrate(tan(x)**3/(a+a*cos(x)),x)
[Out] Integral(tan(x)**3/(cos(x) + 1), x)/a
```

Maxima [A] (verification not implemented)

none

Time = 0.21 (sec) , antiderivative size = 15, normalized size of antiderivative = 0.79

$$\int \frac{\tan^3(x)}{a+a\cos(x)}\,dx = -\frac{2\,\cos{(x)} - 1}{2\,a\cos{(x)}^2}$$

```
[In] integrate(tan(x)^3/(a+a*cos(x)),x, algorithm="maxima")
[Out] -1/2*(2*cos(x) - 1)/(a*cos(x)^2)
```

Giac [A] (verification not implemented)

none

Time = 0.32 (sec) , antiderivative size = 15, normalized size of antiderivative = 0.79

$$\int \frac{\tan^3(x)}{a + a\cos(x)}\, dx = -\frac{2\cos(x) - 1}{2\,a\cos(x)^2}$$

```
[In] integrate(tan(x)^3/(a+a*cos(x)),x, algorithm="giac")
[Out] -1/2*(2*cos(x) - 1)/(a*cos(x)^2)
```

Mupad [B] (verification not implemented)

Time = 13.48 (sec) , antiderivative size = 13, normalized size of antiderivative = 0.68

$$\int \frac{\tan^3(x)}{a + a\cos(x)}\, dx = -\frac{\cos(x) - \frac{1}{2}}{a\cos(x)^2}$$

```
[In] int(tan(x)^3/(a + a*cos(x)),x)
[Out] -(cos(x) - 1/2)/(a*cos(x)^2)
```

3.3 $\int \frac{\tan^2(x)}{a+a\cos(x)}\,dx$

Optimal result

Integrand size = 13, antiderivative size = 15

$$\int \frac{\tan^2(x)}{a+a\cos(x)}\,dx=-\frac{\operatorname{arctanh}(\sin(x))}{a}+\frac{\tan(x)}{a}$$

```
[Out] -arctanh(sin(x))/a+tan(x)/a
```

Rubi [A] (verified)

Time = 0.06 (sec) , antiderivative size = 15, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.308, Rules used = {2785, 3852, 8, 3855}

$$\int \frac{\tan^2(x)}{a+a\cos(x)}\,dx=\frac{\tan(x)}{a}-\frac{\operatorname{arctanh}(\sin(x))}{a}$$

```
[In] Int[Tan[x]^2/(a + a*Cos[x]),x]
```

```
[Out] -(ArcTanh[Sin[x]]/a) + Tan[x]/a
```

Rule 8

```
Int[a_, x_Symbol] :> Simp[a*x, x] /; FreeQ[a, x]
```

Rule 2785

```
Int[((g_.)*tan[(e_.) + (f_.)*(x_)])^(p_.)/((a_) + (b_.)*sin[(e_.) + (f_.)*(
x_)]), x_Symbol] :> Dist[1/a, Int[Sec[e + f*x]^2*(g*Tan[e + f*x])^p, x], x]
 - Dist[1/(b*g), Int[Sec[e + f*x]*(g*Tan[e + f*x])^(p + 1), x], x] /; FreeQ
[{a, b, e, f, g, p}, x] && EqQ[a^2 - b^2, 0] && NeQ[p, -1]
```

Rule 3852

```
Int[csc[(c_.) + (d_.)*(x_)]^(n_), x_Symbol] :> Dist[-d^(-1), Subst[Int[Expa
ndIntegrand[(1 + x^2)^(n/2 - 1), x], x], x, Cot[c + d*x]], x] /; FreeQ[{c,
d}, x] && IGtQ[n/2, 0]
```

Rule 3855

```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] :> Simp[-ArcTanh[Cos[c + d*x]]/d, x]
 /; FreeQ[{c, d}, x]
```

Rubi steps

$$\begin{aligned}\text{integral} &= -\frac{\int \sec(x)\,dx}{a} + \frac{\int \sec^2(x)\,dx}{a}\\ &= -\frac{\operatorname{arctanh}(\sin(x))}{a} - \frac{\operatorname{Subst}(\int 1\,dx, x, -\tan(x))}{a}\\ &= -\frac{\operatorname{arctanh}(\sin(x))}{a} + \frac{\tan(x)}{a}\end{aligned}$$

Mathematica [B] (verified)

Leaf count is larger than twice the leaf count of optimal. 39 vs. $2(15) = 30$.

Time = 0.13 (sec) , antiderivative size = 39, normalized size of antiderivative = 2.60

$$\int \frac{\tan^2(x)}{a + a\cos(x)}\,dx = \frac{\log\left(\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\right) - \log\left(\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)\right) + \tan(x)}{a}$$

```
[In] Integrate[Tan[x]^2/(a + a*Cos[x]),x]
[Out] (Log[Cos[x/2] - Sin[x/2]] - Log[Cos[x/2] + Sin[x/2]] + Tan[x])/a
```

Maple [C] (verified)

Result contains complex when optimal does not.

Time = 0.51 (sec) , antiderivative size = 44, normalized size of antiderivative = 2.93

method	result	size
risch	$\frac{2i}{a(e^{2ix}+1)} - \frac{\ln(e^{ix}+i)}{a} + \frac{\ln(e^{ix}-i)}{a}$	44
default	$\frac{-\frac{1}{\tan(\frac{x}{2})+1} - \ln(\tan(\frac{x}{2})+1) - \frac{1}{\tan(\frac{x}{2})-1} + \ln(\tan(\frac{x}{2})-1)}{a}$	45

```
[In] int(tan(x)^2/(a+cos(x)*a),x,method=_RETURNVERBOSE)
```

```
[Out] 2*I/a/(exp(2*I*x)+1)-1/a*ln(exp(I*x)+I)+1/a*ln(exp(I*x)-I)
```

Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 33 vs. $2(15) = 30$.

Time = 0.33 (sec) , antiderivative size = 33, normalized size of antiderivative = 2.20

$$\int \frac{\tan^2(x)}{a + a\cos(x)}\,dx = -\frac{\cos(x)\log(\sin(x) + 1) - \cos(x)\log(-\sin(x) + 1) - 2\sin(x)}{2\,a\cos(x)}$$

```
[In] integrate(tan(x)^2/(a+a*cos(x)),x, algorithm="fricas")
[Out] -1/2*(cos(x)*log(sin(x) + 1) - cos(x)*log(-sin(x) + 1) - 2*sin(x))/(a*cos(x
))
```

Sympy [F]

$$\int \frac{\tan^2(x)}{a + a\cos(x)}\,dx = \frac{\int \frac{\tan^2(x)}{\cos(x)+1}\,dx}{a}$$

```
[In] integrate(tan(x)**2/(a+a*cos(x)),x)
[Out] Integral(tan(x)**2/(cos(x) + 1), x)/a
```

Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 61 vs. $2(15) = 30$.

Time = 0.21 (sec) , antiderivative size = 61, normalized size of antiderivative = 4.07

$$\int \frac{\tan^2(x)}{a + a\cos(x)}\,dx = -\frac{\log\left(\frac{\sin(x)}{\cos(x)+1} + 1\right)}{a} + \frac{\log\left(\frac{\sin(x)}{\cos(x)+1} - 1\right)}{a} + \frac{2\sin(x)}{\left(a - \frac{a\sin(x)^2}{(\cos(x)+1)^2}\right)(\cos(x) + 1)}$$

```
[In] integrate(tan(x)^2/(a+a*cos(x)),x, algorithm="maxima")
[Out] -log(sin(x)/(cos(x) + 1) + 1)/a + log(sin(x)/(cos(x) + 1) - 1)/a + 2*sin(x)
/((a - a*sin(x)^2/(cos(x) + 1)^2)*(cos(x) + 1))
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 45 vs. $2(15) = 30$.

Time = 0.33 (sec) , antiderivative size = 45, normalized size of antiderivative = 3.00

$$\int \frac{\tan^2(x)}{a + a\cos(x)}\, dx = -\frac{\log\left(\left|\tan\left(\frac{1}{2}\,x\right) + 1\right|\right)}{a} + \frac{\log\left(\left|\tan\left(\frac{1}{2}\,x\right) - 1\right|\right)}{a} - \frac{2\,\tan\left(\frac{1}{2}\,x\right)}{\left(\tan\left(\frac{1}{2}\,x\right)^2 - 1\right)a}$$

```
[In] integrate(tan(x)^2/(a+a*cos(x)),x, algorithm="giac")
[Out] -log(abs(tan(1/2*x) + 1))/a + log(abs(tan(1/2*x) - 1))/a - 2*tan(1/2*x)/((t
an(1/2*x)^2 - 1)*a)
```

Mupad [B] (verification not implemented)

Time = 13.53 (sec) , antiderivative size = 30, normalized size of antiderivative = 2.00

$$\int \frac{\tan^2(x)}{a + a\cos(x)}\, dx = -\frac{2\,\mathrm{atanh}\left(\tan\left(\frac{x}{2}\right)\right)}{a} - \frac{2\tan\left(\frac{x}{2}\right)}{a\,\left(\tan\left(\frac{x}{2}\right)^2 - 1\right)}$$

```
[In] int(tan(x)^2/(a + a*cos(x)),x)
[Out] - (2*atanh(tan(x/2)))/a - (2*tan(x/2))/(a*(tan(x/2)^2 - 1))
```

3.4 $\int \frac{\tan(x)}{a+a\cos(x)}\,dx$

Optimal result

Integrand size = 11, antiderivative size = 18

$$\int \frac{\tan(x)}{a+a\cos(x)}\,dx=-\frac{\log(\cos(x))}{a}+\frac{\log(1+\cos(x))}{a}$$

```
[Out] -ln(cos(x))/a+ln(1+cos(x))/a
```

Rubi [A] (verified)

Time = 0.04 (sec) , antiderivative size = 18, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.364, Rules used = {2786, 36, 29, 31}

$$\int \frac{\tan(x)}{a+a\cos(x)}\,dx=\frac{\log(\cos(x)+1)}{a}-\frac{\log(\cos(x))}{a}$$

```
[In] Int[Tan[x]/(a + a*Cos[x]),x]
```

```
[Out] -(Log[Cos[x]]/a) + Log[1 + Cos[x]]/a
```

Rule 29

```
Int[(x_)^(-1), x_Symbol] :> Simp[Log[x], x]
```

Rule 31

```
Int[((a_) + (b_.)*(x_))^(-1), x_Symbol] :> Simp[Log[RemoveContent[a + b*x,
x]]/b, x] /; FreeQ[{a, b}, x]
```

Rule 36

```
Int[1/(((a_.) + (b_.)*(x_))*((c_.) + (d_.)*(x_))), x_Symbol] :> Dist[b/(b*c
 - a*d), Int[1/(a + b*x), x], x] - Dist[d/(b*c - a*d), Int[1/(c + d*x), x],
 x] /; FreeQ[{a, b, c, d}, x] && NeQ[b*c - a*d, 0]
```

Rule 2786

```
Int[((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m_.)*tan[(e_.) + (f_.)*(x_)]^(p
_.), x_Symbol] :> Dist[1/f, Subst[Int[x^p*((a + x)^(m - (p + 1)/2)/(a - x)^
((p + 1)/2)), x], x, b*Sin[e + f*x]], x] /; FreeQ[{a, b, e, f, m}, x] && Eq
Q[a^2 - b^2, 0] && IntegerQ[(p + 1)/2]
```

Rubi steps

$$
\begin{aligned}
\text{integral} &= -\text{Subst}\left(\int \frac{1}{x(a+x)}\,dx, x, a\cos(x)\right)\\
&= -\frac{\text{Subst}\left(\int \frac{1}{x}\,dx, x, a\cos(x)\right)}{a}+\frac{\text{Subst}\left(\int \frac{1}{a+x}\,dx, x, a\cos(x)\right)}{a}\\
&= -\frac{\log(\cos(x))}{a}+\frac{\log(1+\cos(x))}{a}
\end{aligned}
$$

Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 12, normalized size of antiderivative = 0.67

$$\int \frac{\tan(x)}{a+a\cos(x)}\,dx = \frac{2\text{arctanh}(1+2\cos(x))}{a}$$

```
[In] Integrate[Tan[x]/(a + a*Cos[x]),x]
[Out] (2*ArcTanh[1 + 2*Cos[x]])/a
```

Maple [A] (verified)

Time = 0.61 (sec) , antiderivative size = 16, normalized size of antiderivative = 0.89

method	result	size
default	$\frac{\ln(\cos(x)+1)-\ln(\cos(x))}{a}$	16
risch	$\frac{2\ln(e^{ix}+1)}{a}-\frac{\ln(e^{2ix}+1)}{a}$	28

```
[In] int(tan(x)/(a+cos(x)*a),x,method=_RETURNVERBOSE)
[Out] 1/a*(ln(cos(x)+1)-ln(cos(x)))
```

Fricas [A] (verification not implemented)

none

Time = 0.28 (sec) , antiderivative size = 20, normalized size of antiderivative = 1.11

$$\int \frac{\tan(x)}{a+a\cos(x)}\,dx = -\frac{\log\left(-\cos\left(x\right)\right) - \log\left(\frac{1}{2}\,\cos\left(x\right) + \frac{1}{2}\right)}{a}$$

```
[In] integrate(tan(x)/(a+a*cos(x)),x, algorithm="fricas")
[Out] -(log(-cos(x)) - log(1/2*cos(x) + 1/2))/a
```

Sympy [F]

$$\int \frac{\tan(x)}{a+a\cos(x)}\,dx = \frac{\int \frac{\tan\left(x\right)}{\cos\left(x\right)+1}\,dx}{a}$$

```
[In] integrate(tan(x)/(a+a*cos(x)),x)
[Out] Integral(tan(x)/(cos(x) + 1), x)/a
```

Maxima [A] (verification not implemented)

none

Time = 0.23 (sec) , antiderivative size = 18, normalized size of antiderivative = 1.00

$$\int \frac{\tan(x)}{a+a\cos(x)}\,dx = \frac{\log\left(\cos\left(x\right) + 1\right)}{a} - \frac{\log\left(\cos\left(x\right)\right)}{a}$$

```
[In] integrate(tan(x)/(a+a*cos(x)),x, algorithm="maxima")
[Out] log(cos(x) + 1)/a - log(cos(x))/a
```

Giac [A] (verification not implemented)

none

Time = 0.30 (sec) , antiderivative size = 19, normalized size of antiderivative = 1.06

$$\int \frac{\tan(x)}{a+a\cos(x)}\,dx = \frac{\log\left(\cos\left(x\right) + 1\right)}{a} - \frac{\log\left(|\cos\left(x\right)|\right)}{a}$$

```
[In] integrate(tan(x)/(a+a*cos(x)),x, algorithm="giac")
[Out] log(cos(x) + 1)/a - log(abs(cos(x)))/a
```

Mupad [B] (verification not implemented)

Time = 13.73 (sec) , antiderivative size = 14, normalized size of antiderivative = 0.78

$$\int \frac{\tan(x)}{a+a\cos(x)}\,dx = -\frac{\ln\left(\tan\left(\frac{x}{2}\right)^2-1\right)}{a}$$

```
[In] int(tan(x)/(a + a*cos(x)),x)
[Out] -log(tan(x/2)^2 - 1)/a
```

3.5 $\int \frac{\cot(x)}{a+a\cos(x)}\,dx$

Optimal result

Integrand size = 11, antiderivative size = 33

$$\int \frac{\cot(x)}{a+a\cos(x)}\,dx = -\frac{\operatorname{arctanh}(\cos(x))}{2a}+\frac{\cot(x)\csc(x)}{2a}-\frac{\csc^2(x)}{2a}$$

```
[Out] -1/2*arctanh(cos(x))/a+1/2*cot(x)*csc(x)/a-1/2*csc(x)^2/a
```

Rubi [A] (verified)

Time = 0.06 (sec) , antiderivative size = 33, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.455, Rules used = {2785, 2686, 30, 2691, 3855}

$$\int \frac{\cot(x)}{a+a\cos(x)}\,dx = -\frac{\operatorname{arctanh}(\cos(x))}{2a}-\frac{\csc^2(x)}{2a}+\frac{\cot(x)\csc(x)}{2a}$$

```
[In] Int[Cot[x]/(a + a*Cos[x]),x]
```

```
[Out] -1/2*ArcTanh[Cos[x]]/a + (Cot[x]*Csc[x])/(2*a) - Csc[x]^2/(2*a)
```

Rule 30

```
Int[(x_)^(m_.), x_Symbol] :> Simp[x^(m + 1)/(m + 1), x] /; FreeQ[m, x] && N
eQ[m, -1]
```

Rule 2686

```
Int[((a_.)*sec[(e_.) + (f_.)*(x_)])^(m_.)*((b_.)*tan[(e_.) + (f_.)*(x_)])^(
n_.), x_Symbol] :> Dist[a/f, Subst[Int[(a*x)^(m - 1)*(-1 + x^2)^((n - 1)/2)
, x], x, Sec[e + f*x]], x] /; FreeQ[{a, e, f, m}, x] && IntegerQ[(n - 1)/2]
```

```
 &&  !(IntegerQ[m/2] && LtQ[0, m, n + 1])
```

Rule 2691

```
Int[((a_.)*sec[(e_.) + (f_.)*(x_)])^(m_.)*((b_.)*tan[(e_.) + (f_.)*(x_)])^(
n_), x_Symbol] :> Simp[b*(a*Sec[e + f*x])^m*((b*Tan[e + f*x])^(n - 1)/(f*(m
 + n - 1))), x] - Dist[b^2*((n - 1)/(m + n - 1)), Int[(a*Sec[e + f*x])^m*(b
*Tan[e + f*x])^(n - 2), x], x] /; FreeQ[{a, b, e, f, m}, x] && GtQ[n, 1] &&
 NeQ[m + n - 1, 0] && IntegersQ[2*m, 2*n]
```

Rule 2785

```
Int[((g_.)*tan[(e_.) + (f_.)*(x_)])^(p_.)/((a_) + (b_.)*sin[(e_.) + (f_.)*(
x_)]), x_Symbol] :> Dist[1/a, Int[Sec[e + f*x]^2*(g*Tan[e + f*x])^p, x], x]
 - Dist[1/(b*g), Int[Sec[e + f*x]*(g*Tan[e + f*x])^(p + 1), x], x] /; FreeQ
[{a, b, e, f, g, p}, x] && EqQ[a^2 - b^2, 0] && NeQ[p, -1]
```

Rule 3855

```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] :> Simp[-ArcTanh[Cos[c + d*x]]/d, x]
 /; FreeQ[{c, d}, x]
```

Rubi steps

$$
\begin{aligned}
\text{integral} &= -\frac{\int \cot^2(x)\csc(x)\,dx}{a} + \frac{\int \cot(x)\csc^2(x)\,dx}{a} \\
&= \frac{\cot(x)\csc(x)}{2a} + \frac{\int \csc(x)\,dx}{2a} - \frac{\operatorname{Subst}(\int x\,dx, x, \csc(x))}{a} \\
&= -\frac{\operatorname{arctanh}(\cos(x))}{2a} + \frac{\cot(x)\csc(x)}{2a} - \frac{\csc^2(x)}{2a}
\end{aligned}
$$

Mathematica [A] (verified)

Time = 0.03 (sec) , antiderivative size = 42, normalized size of antiderivative = 1.27

$$
\int \frac{\cot(x)}{a + a\cos(x)}\,dx = -\frac{1 + 2\cos^2\left(\frac{x}{2}\right)\left(\log\left(\cos\left(\frac{x}{2}\right)\right) - \log\left(\sin\left(\frac{x}{2}\right)\right)\right)}{2a(1 + \cos(x))}
$$

```
[In] Integrate[Cot[x]/(a + a*Cos[x]),x]
[Out] -1/2*(1 + 2*Cos[x/2]^2*(Log[Cos[x/2]] - Log[Sin[x/2]]))/(a*(1 + Cos[x]))
```

Maple [A] (verified)

Time = 0.51 (sec) , antiderivative size = 28, normalized size of antiderivative = 0.85

method	result	size
default	$\frac{-\frac{1}{2(\cos(x)+1)}-\frac{\ln(\cos(x)+1)}{4}+\frac{\ln(\cos(x)-1)}{4}}{a}$	28
risch	$-\frac{e^{ix}}{(e^{ix}+1)^2 a}+\frac{\ln(e^{ix}-1)}{2a}-\frac{\ln(e^{ix}+1)}{2a}$	47

```
[In] int(cot(x)/(a+cos(x)*a),x,method=_RETURNVERBOSE)
[Out] 1/a*(-1/2/(cos(x)+1)-1/4*ln(cos(x)+1)+1/4*ln(cos(x)-1))
```

Fricas [A] (verification not implemented)

none

Time = 0.27 (sec) , antiderivative size = 37, normalized size of antiderivative = 1.12

$$\int \frac{\cot(x)}{a+a\cos(x)}\,dx = -\frac{(\cos(x)+1)\log\left(\frac{1}{2}\cos(x)+\frac{1}{2}\right)-(\cos(x)+1)\log\left(-\frac{1}{2}\cos(x)+\frac{1}{2}\right)+2}{4\,(a\cos(x)+a)}$$

```
[In] integrate(cot(x)/(a+a*cos(x)),x, algorithm="fricas")
[Out] -1/4*((cos(x) + 1)*log(1/2*cos(x) + 1/2) - (cos(x) + 1)*log(-1/2*cos(x) + 1
/2) + 2)/(a*cos(x) + a)
```

Sympy [F]

$$\int \frac{\cot(x)}{a+a\cos(x)}\,dx = \frac{\int \frac{\cot(x)}{\cos(x)+1}\,dx}{a}$$

```
[In] integrate(cot(x)/(a+a*cos(x)),x)
[Out] Integral(cot(x)/(cos(x) + 1), x)/a
```

Maxima [A] (verification not implemented)

none

Time = 0.23 (sec) , antiderivative size = 31, normalized size of antiderivative = 0.94

$$\int \frac{\cot(x)}{a + a\cos(x)}\, dx = -\frac{\log\left(\cos\left(x\right) + 1\right)}{4\,a} + \frac{\log\left(\cos\left(x\right) - 1\right)}{4\,a} - \frac{1}{2\,\left(a\cos\left(x\right) + a\right)}$$

```
[In] integrate(cot(x)/(a+a*cos(x)),x, algorithm="maxima")
[Out] -1/4*log(cos(x) + 1)/a + 1/4*log(cos(x) - 1)/a - 1/2/(a*cos(x) + a)
```

Giac [A] (verification not implemented)

none

Time = 0.30 (sec) , antiderivative size = 34, normalized size of antiderivative = 1.03

$$\int \frac{\cot(x)}{a + a\cos(x)}\, dx = -\frac{\log\left(\cos\left(x\right) + 1\right)}{4\,a} + \frac{\log\left(-\cos\left(x\right) + 1\right)}{4\,a} - \frac{1}{2\,a(\cos\left(x\right) + 1)}$$

```
[In] integrate(cot(x)/(a+a*cos(x)),x, algorithm="giac")
[Out] -1/4*log(cos(x) + 1)/a + 1/4*log(-cos(x) + 1)/a - 1/2/(a*(cos(x) + 1))
```

Mupad [B] (verification not implemented)

Time = 14.07 (sec) , antiderivative size = 21, normalized size of antiderivative = 0.64

$$\int \frac{\cot(x)}{a + a\cos(x)}\, dx = \frac{2\,\ln\left(\tan\left(\frac{x}{2}\right)\right) - {\tan\left(\frac{x}{2}\right)}^2}{4\,a}$$

```
[In] int(cot(x)/(a + a*cos(x)),x)
[Out] (2*log(tan(x/2)) - tan(x/2)^2)/(4*a)
```

3.6 $\int \frac{\cot^2(x)}{a+a\cos(x)}\,dx$

Optimal result

Integrand size = 13, antiderivative size = 30

$$\int \frac{\cot^2(x)}{a+a\cos(x)}\,dx=-\frac{\cot^3(x)}{3a}-\frac{\csc(x)}{a}+\frac{\csc^3(x)}{3a}$$

```
[Out] -1/3*cot(x)^3/a-csc(x)/a+1/3*csc(x)^3/a
```

Rubi [A] (verified)

Time = 0.10 (sec) , antiderivative size = 30, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 4, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.308, Rules used = {2785, 2687, 30, 2686}

$$\int \frac{\cot^2(x)}{a+a\cos(x)}\,dx=-\frac{\cot^3(x)}{3a}+\frac{\csc^3(x)}{3a}-\frac{\csc(x)}{a}$$

```
[In] Int[Cot[x]^2/(a + a*Cos[x]),x]
```

```
[Out] -1/3*Cot[x]^3/a - Csc[x]/a + Csc[x]^3/(3*a)
```

Rule 30

```
Int[(x_)^(m_.), x_Symbol] :> Simp[x^(m + 1)/(m + 1), x] /; FreeQ[m, x] && N
eQ[m, -1]
```

Rule 2686

```
Int[((a_.)*sec[(e_.) + (f_.)*(x_)])^(m_.)*((b_.)*tan[(e_.) + (f_.)*(x_)])^(
n_.), x_Symbol] :> Dist[a/f, Subst[Int[(a*x)^(m - 1)*(-1 + x^2)^((n - 1)/2)
, x], x, Sec[e + f*x]], x] /; FreeQ[{a, e, f, m}, x] && IntegerQ[(n - 1)/2]
```

```
&& !(IntegerQ[m/2] && LtQ[0, m, n + 1])
```

Rule 2687

```
Int[sec[(e_.) + (f_.)*(x_)]^(m_)*((b_.)*tan[(e_.) + (f_.)*(x_)])^(n_.), x_S
ymbol] :> Dist[1/f, Subst[Int[(b*x)^n*(1 + x^2)^(m/2 - 1), x], x, Tan[e + f
*x]], x] /; FreeQ[{b, e, f, n}, x] && IntegerQ[m/2] && !(IntegerQ[(n - 1)/
2] && LtQ[0, n, m - 1])
```

Rule 2785

```
Int[((g_.)*tan[(e_.) + (f_.)*(x_)])^(p_.)/((a_) + (b_.)*sin[(e_.) + (f_.)*(
x_)]), x_Symbol] :> Dist[1/a, Int[Sec[e + f*x]^2*(g*Tan[e + f*x])^p, x], x]
 - Dist[1/(b*g), Int[Sec[e + f*x]*(g*Tan[e + f*x])^(p + 1), x], x] /; FreeQ
[{a, b, e, f, g, p}, x] && EqQ[a^2 - b^2, 0] && NeQ[p, -1]
```

Rubi steps

$$\begin{aligned}\text{integral} &= -\frac{\int \cot^3(x)\csc(x)\,dx}{a} + \frac{\int \cot^2(x)\csc^2(x)\,dx}{a}\\ &= \frac{\text{Subst}\left(\int x^2\,dx, x, -\cot(x)\right)}{a} + \frac{\text{Subst}\left(\int \left(-1+x^2\right)\,dx, x, \csc(x)\right)}{a}\\ &= -\frac{\cot^3(x)}{3a} - \frac{\csc(x)}{a} + \frac{\csc^3(x)}{3a}\end{aligned}$$

Mathematica [A] (verified)

Time = 0.13 (sec) , antiderivative size = 25, normalized size of antiderivative = 0.83

$$\int \frac{\cot^2(x)}{a+a\cos(x)}\,dx = \frac{(-3-4\cos(x)+\cos(2x))\csc(x)}{6a(1+\cos(x))}$$

```
[In] Integrate[Cot[x]^2/(a + a*Cos[x]),x]
[Out] ((-3 - 4*Cos[x] + Cos[2*x])*Csc[x])/(6*a*(1 + Cos[x]))
```

Maple [A] (verified)

Time = 0.46 (sec) , antiderivative size = 29, normalized size of antiderivative = 0.97

method	result	size
default	$\frac{\frac{\left(\tan^{3}\left(\frac{x}{2}\right)\right)}{3}-2\tan\left(\frac{x}{2}\right)-\frac{1}{\tan\left(\frac{x}{2}\right)}}{4a}$	29
risch	$-\frac{2i\left(3\,\mathrm{e}^{3ix}+3\,\mathrm{e}^{2ix}+\mathrm{e}^{ix}-1\right)}{3\left(\mathrm{e}^{ix}+1\right)^{3}a\left(\mathrm{e}^{ix}-1\right)}$	46

```
[In] int(cot(x)^2/(a+cos(x)*a),x,method=_RETURNVERBOSE)
[Out] 1/4/a*(1/3*tan(1/2*x)^3-2*tan(1/2*x)-1/tan(1/2*x))
```

Fricas [A] (verification not implemented)

none

Time = 0.28 (sec) , antiderivative size = 24, normalized size of antiderivative = 0.80

$$\int \frac{\cot^2(x)}{a+a\cos(x)}\,dx = \frac{\cos\left(x\right)^{2}-2\,\cos\left(x\right)-2}{3\left(a\cos\left(x\right)+a\right)\sin\left(x\right)}$$

```
[In] integrate(cot(x)^2/(a+a*cos(x)),x, algorithm="fricas")
[Out] 1/3*(cos(x)^2 - 2*cos(x) - 2)/((a*cos(x) + a)*sin(x))
```

Sympy [F]

$$\int \frac{\cot^2(x)}{a+a\cos(x)}\,dx = \frac{\int \frac{\cot^{2}(x)}{\cos(x)+1}\,dx}{a}$$

```
[In] integrate(cot(x)**2/(a+a*cos(x)),x)
[Out] Integral(cot(x)**2/(cos(x) + 1), x)/a
```

Maxima [A] (verification not implemented)

none

Time = 0.22 (sec) , antiderivative size = 42, normalized size of antiderivative = 1.40

$$\int \frac{\cot^2(x)}{a + a\cos(x)}\,dx = -\frac{\frac{6\,\sin(x)}{\cos(x)+1} - \frac{\sin(x)^3}{(\cos(x)+1)^3}}{12\,a} - \frac{\cos(x) + 1}{4\,a\sin(x)}$$

```
[In] integrate(cot(x)^2/(a+a*cos(x)),x, algorithm="maxima")
[Out] -1/12*(6*sin(x)/(cos(x) + 1) - sin(x)^3/(cos(x) + 1)^3)/a - 1/4*(cos(x) + 1
)/(a*sin(x))
```

Giac [A] (verification not implemented)

none

Time = 0.31 (sec) , antiderivative size = 37, normalized size of antiderivative = 1.23

$$\int \frac{\cot^2(x)}{a + a\cos(x)}\,dx = \frac{a^2\tan\left(\frac{1}{2}\,x\right)^3 - 6\,a^2\tan\left(\frac{1}{2}\,x\right)}{12\,a^3} - \frac{1}{4\,a\tan\left(\frac{1}{2}\,x\right)}$$

```
[In] integrate(cot(x)^2/(a+a*cos(x)),x, algorithm="giac")
[Out] 1/12*(a^2*tan(1/2*x)^3 - 6*a^2*tan(1/2*x))/a^3 - 1/4/(a*tan(1/2*x))
```

Mupad [B] (verification not implemented)

Time = 14.04 (sec) , antiderivative size = 35, normalized size of antiderivative = 1.17

$$\int \frac{\cot^2(x)}{a + a\cos(x)}\,dx = \frac{4\cos\left(\frac{x}{2}\right)^4 - 8\cos\left(\frac{x}{2}\right)^2 + 1}{12\,a\cos\left(\frac{x}{2}\right)^3\,\sin\left(\frac{x}{2}\right)}$$

```
[In] int(cot(x)^2/(a + a*cos(x)),x)
[Out] (4*cos(x/2)^4 - 8*cos(x/2)^2 + 1)/(12*a*cos(x/2)^3*sin(x/2))
```

3.7 $\int \frac{\cot^3(x)}{a+a\cos(x)}\,dx$

Optimal result

Integrand size = 13, antiderivative size = 46

$$\int \frac{\cot^3(x)}{a+a\cos(x)}\,dx=\frac{3\text{arctanh}(\cos(x))}{8a}-\frac{\cot^4(x)}{4a}-\frac{3\cot(x)\csc(x)}{8a}+\frac{\cot^3(x)\csc(x)}{4a}$$

```
[Out] 3/8*arctanh(cos(x))/a-1/4*cot(x)^4/a-3/8*cot(x)*csc(x)/a+1/4*cot(x)^3*csc(x
)/a
```

Rubi [A] (verified)

Time = 0.14 (sec) , antiderivative size = 46, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.385, Rules used = {2785, 2687, 30, 2691, 3855}

$$\int \frac{\cot^3(x)}{a+a\cos(x)}\,dx=\frac{3\text{arctanh}(\cos(x))}{8a}-\frac{\cot^4(x)}{4a}+\frac{\cot^3(x)\csc(x)}{4a}-\frac{3\cot(x)\csc(x)}{8a}$$

```
[In] Int[Cot[x]^3/(a + a*Cos[x]),x]
```

```
[Out] (3*ArcTanh[Cos[x]])/(8*a) - Cot[x]^4/(4*a) - (3*Cot[x]*Csc[x])/(8*a) + (Cot
[x]^3*Csc[x])/(4*a)
```

Rule 30

```
Int[(x_)^(m_.), x_Symbol] :> Simp[x^(m + 1)/(m + 1), x] /; FreeQ[m, x] && N
eQ[m, -1]
```

Rule 2687

```
Int[sec[(e_.) + (f_.)*(x_)]^(m_)*((b_.)*tan[(e_.) + (f_.)*(x_)])^(n_.), x_S
ymbol] :> Dist[1/f, Subst[Int[(b*x)^n*(1 + x^2)^(m/2 - 1), x], x, Tan[e + f
```

```
*x]], x] /; FreeQ[{b, e, f, n}, x] && IntegerQ[m/2] &&  !(IntegerQ[(n - 1)/
2] && LtQ[0, n, m - 1])
```

Rule 2691

```
Int[((a_.)*sec[(e_.) + (f_.)*(x_)])^(m_.)*((b_.)*tan[(e_.) + (f_.)*(x_)])^(
n_), x_Symbol] :> Simp[b*(a*Sec[e + f*x])^m*((b*Tan[e + f*x])^(n - 1)/(f*(m
 + n - 1))), x] - Dist[b^2*((n - 1)/(m + n - 1)), Int[(a*Sec[e + f*x])^m*(b
*Tan[e + f*x])^(n - 2), x], x] /; FreeQ[{a, b, e, f, m}, x] && GtQ[n, 1] &&
 NeQ[m + n - 1, 0] && IntegersQ[2*m, 2*n]
```

Rule 2785

```
Int[((g_.)*tan[(e_.) + (f_.)*(x_)])^(p_.)/((a_) + (b_.)*sin[(e_.) + (f_.)*(
x_)]), x_Symbol] :> Dist[1/a, Int[Sec[e + f*x]^2*(g*Tan[e + f*x])^p, x], x]
 - Dist[1/(b*g), Int[Sec[e + f*x]*(g*Tan[e + f*x])^(p + 1), x], x] /; FreeQ
[{a, b, e, f, g, p}, x] && EqQ[a^2 - b^2, 0] && NeQ[p, -1]
```

Rule 3855

```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] :> Simp[-ArcTanh[Cos[c + d*x]]/d, x]
 /; FreeQ[{c, d}, x]
```

Rubi steps

$$\begin{aligned}\text{integral} &= -\frac{\int \cot^4(x)\csc(x)\,dx}{a}+\frac{\int \cot^3(x)\csc^2(x)\,dx}{a}\\ &= \frac{\cot^3(x)\csc(x)}{4a}+\frac{3\int \cot^2(x)\csc(x)\,dx}{4a}-\frac{\text{Subst}\left(\int x^3\,dx,x,-\cot(x)\right)}{a}\\ &= -\frac{\cot^4(x)}{4a}-\frac{3\cot(x)\csc(x)}{8a}+\frac{\cot^3(x)\csc(x)}{4a}-\frac{3\int \csc(x)\,dx}{8a}\\ &= \frac{3\text{arctanh}(\cos(x))}{8a}-\frac{\cot^4(x)}{4a}-\frac{3\cot(x)\csc(x)}{8a}+\frac{\cot^3(x)\csc(x)}{4a}\end{aligned}$$

Mathematica [A] (verified)

Time = 0.10 (sec) , antiderivative size = 60, normalized size of antiderivative = 1.30

$$\begin{aligned}&\int \frac{\cot^3(x)}{a+a\cos(x)}\,dx\\ &= -\frac{-8+2\cot^2\left(\frac{x}{2}\right)-12\cos^2\left(\frac{x}{2}\right)\left(\log\left(\cos\left(\frac{x}{2}\right)\right)-\log\left(\sin\left(\frac{x}{2}\right)\right)\right)+\sec^2\left(\frac{x}{2}\right)}{16a(1+\cos(x))}\end{aligned}$$

```
[In] Integrate[Cot[x]^3/(a + a*Cos[x]),x]
[Out] -1/16*(-8 + 2*Cot[x/2]^2 - 12*Cos[x/2]^2*(Log[Cos[x/2]] - Log[Sin[x/2]]) +
Sec[x/2]^2)/(a*(1 + Cos[x]))
```

Maple [A] (verified)

Time = 0.54 (sec) , antiderivative size = 44, normalized size of antiderivative = 0.96

method	result	size
default	$\frac{\frac{1}{8\cos(x)-8}-\frac{3\ln(\cos(x)-1)}{16}-\frac{1}{8(\cos(x)+1)^2}+\frac{1}{2\cos(x)+2}+\frac{3\ln(\cos(x)+1)}{16}}{a}$	44
risch	$\frac{5\,e^{5ix}+2\,e^{4ix}+2\,e^{3ix}+2\,e^{2ix}+5\,e^{ix}}{4(e^{ix}+1)^4 a(e^{ix}-1)^2}-\frac{3\ln(e^{ix}-1)}{8a}+\frac{3\ln(e^{ix}+1)}{8a}$	87

```
[In] int(cot(x)^3/(a+cos(x)*a),x,method=_RETURNVERBOSE)
```

```
[Out] 1/a*(1/8/(cos(x)-1)-3/16*ln(cos(x)-1)-1/8/(cos(x)+1)^2+1/2/(cos(x)+1)+3/16*
ln(cos(x)+1))
```

Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 83 vs. 2(38) = 76.

Time = 0.28 (sec) , antiderivative size = 83, normalized size of antiderivative = 1.80

$$\int \frac{\cot^3(x)}{a+a\cos(x)}\,dx$$
$$=\frac{10\cos(x)^2+3\left(\cos(x)^3+\cos(x)^2-\cos(x)-1\right)\log\left(\frac{1}{2}\cos(x)+\frac{1}{2}\right)-3\left(\cos(x)^3+\cos(x)^2-\cos(x)-\right.}{16\left(a\cos(x)^3+a\cos(x)^2-a\cos(x)-a\right)}$$

```
[In] integrate(cot(x)^3/(a+a*cos(x)),x, algorithm="fricas")
```

```
[Out] 1/16*(10*cos(x)^2 + 3*(cos(x)^3 + cos(x)^2 - cos(x) - 1)*log(1/2*cos(x) + 1
/2) - 3*(cos(x)^3 + cos(x)^2 - cos(x) - 1)*log(-1/2*cos(x) + 1/2) + 2*cos(x
) - 4)/(a*cos(x)^3 + a*cos(x)^2 - a*cos(x) - a)
```

Sympy [F]

$$\int \frac{\cot^3(x)}{a+a\cos(x)}\,dx=\frac{\int \frac{\cot^3(x)}{\cos(x)+1}\,dx}{a}$$

```
[In] integrate(cot(x)**3/(a+a*cos(x)),x)
```

```
[Out] Integral(cot(x)**3/(cos(x) + 1), x)/a
```

Maxima [A] (verification not implemented)

none

Time = 0.24 (sec) , antiderivative size = 56, normalized size of antiderivative = 1.22

$$\int \frac{\cot^3(x)}{a + a\cos(x)}\,dx = \frac{5\,\cos\left(x\right)^2 + \cos\left(x\right) - 2}{8\,\left(a\cos\left(x\right)^3 + a\cos\left(x\right)^2 - a\cos\left(x\right) - a\right)} + \frac{3\,\log\left(\cos\left(x\right) + 1\right)}{16\,a} - \frac{3\,\log\left(\cos\left(x\right) - 1\right)}{16\,a}$$

```
[In] integrate(cot(x)^3/(a+a*cos(x)),x, algorithm="maxima")
```

```
[Out] 1/8*(5*cos(x)^2 + cos(x) - 2)/(a*cos(x)^3 + a*cos(x)^2 - a*cos(x) - a) + 3/
16*log(cos(x) + 1)/a - 3/16*log(cos(x) - 1)/a
```

Giac [A] (verification not implemented)

none

Time = 0.31 (sec) , antiderivative size = 50, normalized size of antiderivative = 1.09

$$\int \frac{\cot^3(x)}{a + a\cos(x)}\,dx = \frac{3\,\log\left(\cos\left(x\right) + 1\right)}{16\,a} - \frac{3\,\log\left(-\cos\left(x\right) + 1\right)}{16\,a} + \frac{5\,\cos\left(x\right)^2 + \cos\left(x\right) - 2}{8\,a(\cos\left(x\right) + 1)^2(\cos\left(x\right) - 1)}$$

```
[In] integrate(cot(x)^3/(a+a*cos(x)),x, algorithm="giac")
```

```
[Out] 3/16*log(cos(x) + 1)/a - 3/16*log(-cos(x) + 1)/a + 1/8*(5*cos(x)^2 + cos(x)
 - 2)/(a*(cos(x) + 1)^2*(cos(x) - 1))
```

Mupad [B] (verification not implemented)

Time = 13.84 (sec) , antiderivative size = 40, normalized size of antiderivative = 0.87

$$\int \frac{\cot^3(x)}{a + a\cos(x)}\,dx = -\frac{\tan\left(\frac{x}{2}\right)^6 - 6\tan\left(\frac{x}{2}\right)^4 + 12\tan\left(\frac{x}{2}\right)^2\,\ln\left(\tan\left(\frac{x}{2}\right)\right) + 2}{32\,a\tan\left(\frac{x}{2}\right)^2}$$

```
[In] int(cot(x)^3/(a + a*cos(x)),x)
```

```
[Out] -(tan(x/2)^6 - 6*tan(x/2)^4 + 12*tan(x/2)^2*log(tan(x/2)) + 2)/(32*a*tan(x/
2)^2)
```

3.8 $\int \frac{\cot^4(x)}{a+a\cos(x)}\,dx$

Optimal result

Integrand size = 13, antiderivative size = 40

$$\int \frac{\cot^4(x)}{a+a\cos(x)}\,dx = -\frac{\cot^5(x)}{5a}+\frac{\csc(x)}{a}-\frac{2\csc^3(x)}{3a}+\frac{\csc^5(x)}{5a}$$

```
[Out] -1/5*cot(x)^5/a+csc(x)/a-2/3*csc(x)^3/a+1/5*csc(x)^5/a
```

Rubi [A] (verified)

Time = 0.10 (sec) , antiderivative size = 40, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.385, Rules used = {2785, 2687, 30, 2686, 200}

$$\int \frac{\cot^4(x)}{a+a\cos(x)}\,dx = -\frac{\cot^5(x)}{5a}+\frac{\csc^5(x)}{5a}-\frac{2\csc^3(x)}{3a}+\frac{\csc(x)}{a}$$

```
[In] Int[Cot[x]^4/(a + a*Cos[x]),x]
```

```
[Out] -1/5*Cot[x]^5/a + Csc[x]/a - (2*Csc[x]^3)/(3*a) + Csc[x]^5/(5*a)
```

Rule 30

```
Int[(x_)^(m_.), x_Symbol] :> Simp[x^(m + 1)/(m + 1), x] /; FreeQ[m, x] && N
eQ[m, -1]
```

Rule 200

```
Int[((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Int[ExpandIntegrand[(a + b*
x^n)^p, x], x] /; FreeQ[{a, b}, x] && IGtQ[n, 0] && IGtQ[p, 0]
```

Rule 2686

```
Int[((a_.)*sec[(e_.) + (f_.)*(x_)])^(m_.)*((b_.)*tan[(e_.) + (f_.)*(x_)])^(
n_.), x_Symbol] :> Dist[a/f, Subst[Int[(a*x)^(m - 1)*(-1 + x^2)^((n - 1)/2)
, x], x, Sec[e + f*x]], x] /; FreeQ[{a, e, f, m}, x] && IntegerQ[(n - 1)/2]
 && !(IntegerQ[m/2] && LtQ[0, m, n + 1])
```

Rule 2687

```
Int[sec[(e_.) + (f_.)*(x_)]^(m_)*((b_.)*tan[(e_.) + (f_.)*(x_)])^(n_.), x_S
ymbol] :> Dist[1/f, Subst[Int[(b*x)^n*(1 + x^2)^(m/2 - 1), x], x, Tan[e + f
*x]], x] /; FreeQ[{b, e, f, n}, x] && IntegerQ[m/2] && !(IntegerQ[(n - 1)/
2] && LtQ[0, n, m - 1])
```

Rule 2785

```
Int[((g_.)*tan[(e_.) + (f_.)*(x_)])^(p_.)/((a_) + (b_.)*sin[(e_.) + (f_.)*(
x_)]), x_Symbol] :> Dist[1/a, Int[Sec[e + f*x]^2*(g*Tan[e + f*x])^p, x], x]
 - Dist[1/(b*g), Int[Sec[e + f*x]*(g*Tan[e + f*x])^(p + 1), x], x] /; FreeQ
[{a, b, e, f, g, p}, x] && EqQ[a^2 - b^2, 0] && NeQ[p, -1]
```

Rubi steps

$$\begin{aligned}
\text{integral} &= -\frac{\int \cot^5(x)\csc(x)\,dx}{a}+\frac{\int \cot^4(x)\csc^2(x)\,dx}{a}\\
&= \frac{\text{Subst}\left(\int x^4\,dx,x,-\cot(x)\right)}{a}+\frac{\text{Subst}\left(\int \left(-1+x^2\right)^2\,dx,x,\csc(x)\right)}{a}\\
&= -\frac{\cot^5(x)}{5a}+\frac{\text{Subst}\left(\int \left(1-2x^2+x^4\right)\,dx,x,\csc(x)\right)}{a}\\
&= -\frac{\cot^5(x)}{5a}+\frac{\csc(x)}{a}-\frac{2\csc^3(x)}{3a}+\frac{\csc^5(x)}{5a}
\end{aligned}$$

Mathematica [A] (verified)

Time = 0.14 (sec) , antiderivative size = 41, normalized size of antiderivative = 1.02

$$\int \frac{\cot^4(x)}{a+a\cos(x)}\,dx=-\frac{(-25+8\cos(x)+36\cos(2x)+24\cos(3x)-3\cos(4x))\csc^3(x)}{120a(1+\cos(x))}$$

```
[In] Integrate[Cot[x]^4/(a + a*Cos[x]),x]
[Out] -1/120*((-25 + 8*Cos[x] + 36*Cos[2*x] + 24*Cos[3*x] - 3*Cos[4*x])*Csc[x]^3)
/(a*(1 + Cos[x]))
```

Maple [A] (verified)

Time = 0.65 (sec) , antiderivative size = 45, normalized size of antiderivative = 1.12

method	result	size
default	$\frac{\frac{\left(\tan^{5}\left(\frac{x}{2}\right)\right)}{5}-\frac{4\left(\tan^{3}\left(\frac{x}{2}\right)\right)}{3}+6\tan\left(\frac{x}{2}\right)-\frac{1}{3\tan\left(\frac{x}{2}\right)^{3}}+\frac{4}{\tan\left(\frac{x}{2}\right)}}{16a}$	45
risch	$\frac{2i\left(15\,\mathrm{e}^{7ix}+15\,\mathrm{e}^{6ix}-5\,\mathrm{e}^{5ix}-25\,\mathrm{e}^{4ix}+13\,\mathrm{e}^{3ix}+21\,\mathrm{e}^{2ix}+9\,\mathrm{e}^{ix}-3\right)}{15\left(\mathrm{e}^{ix}+1\right)^{5}a\left(\mathrm{e}^{ix}-1\right)^{3}}$	76

```
[In] int(cot(x)^4/(a+cos(x)*a),x,method=_RETURNVERBOSE)
```

```
[Out] 1/16/a*(1/5*tan(1/2*x)^5-4/3*tan(1/2*x)^3+6*tan(1/2*x)-1/3/tan(1/2*x)^3+4/t
an(1/2*x))
```

Fricas [A] (verification not implemented)

none

Time = 0.26 (sec) , antiderivative size = 53, normalized size of antiderivative = 1.32

$$\int \frac{\cot^4(x)}{a+a\cos(x)}\,dx=-\frac{3\,\cos\left(x\right)^{4}-12\,\cos\left(x\right)^{3}-12\,\cos\left(x\right)^{2}+8\,\cos\left(x\right)+8}{15\,\left(a\cos\left(x\right)^{3}+a\cos\left(x\right)^{2}-a\cos\left(x\right)-a\right)\sin\left(x\right)}$$

```
[In] integrate(cot(x)^4/(a+a*cos(x)),x, algorithm="fricas")
```

```
[Out] -1/15*(3*cos(x)^4 - 12*cos(x)^3 - 12*cos(x)^2 + 8*cos(x) + 8)/((a*cos(x)^3
+ a*cos(x)^2 - a*cos(x) - a)*sin(x))
```

Sympy [F]

$$\int \frac{\cot^4(x)}{a+a\cos(x)}\,dx=\frac{\int \frac{\cot^{4}\left(x\right)}{\cos\left(x\right)+1}\,dx}{a}$$

```
[In] integrate(cot(x)**4/(a+a*cos(x)),x)
```

```
[Out] Integral(cot(x)**4/(cos(x) + 1), x)/a
```

Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 70 vs. $2(34) = 68$.

Time = 0.22 (sec) , antiderivative size = 70, normalized size of antiderivative = 1.75

$$\int \frac{\cot^4(x)}{a + a\cos(x)}\, dx = \frac{\frac{90 \sin(x)}{\cos(x)+1} - \frac{20 \sin(x)^3}{(\cos(x)+1)^3} + \frac{3 \sin(x)^5}{(\cos(x)+1)^5}}{240\, a} + \frac{\left(\frac{12 \sin(x)^2}{(\cos(x)+1)^2} - 1\right)(\cos(x) + 1)^3}{48\, a \sin(x)^3}$$

```
[In] integrate(cot(x)^4/(a+a*cos(x)),x, algorithm="maxima")
```

```
[Out] 1/240*(90*sin(x)/(cos(x) + 1) - 20*sin(x)^3/(cos(x) + 1)^3 + 3*sin(x)^5/(co
s(x) + 1)^5)/a + 1/48*(12*sin(x)^2/(cos(x) + 1)^2 - 1)*(cos(x) + 1)^3/(a*si
n(x)^3)
```

Giac [A] (verification not implemented)

none

Time = 0.37 (sec) , antiderivative size = 59, normalized size of antiderivative = 1.48

$$\int \frac{\cot^4(x)}{a + a\cos(x)}\, dx = \frac{12 \tan\left(\frac{1}{2}x\right)^2 - 1}{48\, a \tan\left(\frac{1}{2}x\right)^3} + \frac{3\, a^4 \tan\left(\frac{1}{2}x\right)^5 - 20\, a^4 \tan\left(\frac{1}{2}x\right)^3 + 90\, a^4 \tan\left(\frac{1}{2}x\right)}{240\, a^5}$$

```
[In] integrate(cot(x)^4/(a+a*cos(x)),x, algorithm="giac")
```

```
[Out] 1/48*(12*tan(1/2*x)^2 - 1)/(a*tan(1/2*x)^3) + 1/240*(3*a^4*tan(1/2*x)^5 - 2
0*a^4*tan(1/2*x)^3 + 90*a^4*tan(1/2*x))/a^5
```

Mupad [B] (verification not implemented)

Time = 13.40 (sec) , antiderivative size = 45, normalized size of antiderivative = 1.12

$$\int \frac{\cot^4(x)}{a + a\cos(x)}\, dx = \frac{3 \tan\left(\frac{x}{2}\right)^8 - 20 \tan\left(\frac{x}{2}\right)^6 + 90 \tan\left(\frac{x}{2}\right)^4 + 60 \tan\left(\frac{x}{2}\right)^2 - 5}{240\, a \tan\left(\frac{x}{2}\right)^3}$$

```
[In] int(cot(x)^4/(a + a*cos(x)),x)
```

```
[Out] (60*tan(x/2)^2 + 90*tan(x/2)^4 - 20*tan(x/2)^6 + 3*tan(x/2)^8 - 5)/(240*a*t
an(x/2)^3)
```

3.9 $\int \frac{\tan(3x)}{(1+\cos(3x))^2} \, dx$

Optimal result

Integrand size = 13, antiderivative size = 33

$$\int \frac{\tan(3x)}{(1+\cos(3x))^2} \, dx = -\frac{1}{3(1+\cos(3x))} - \frac{1}{3}\log(\cos(3x)) + \frac{1}{3}\log(1+\cos(3x))$$

```
[Out] -1/3/(1+cos(3*x))-1/3*ln(cos(3*x))+1/3*ln(1+cos(3*x))
```

Rubi [A] (verified)

Time = 0.04 (sec) , antiderivative size = 33, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}} = 0.154$, Rules used = {2786, 46}

$$\int \frac{\tan(3x)}{(1+\cos(3x))^2} \, dx = -\frac{1}{3(\cos(3x)+1)} - \frac{1}{3}\log(\cos(3x)) + \frac{1}{3}\log(\cos(3x)+1)$$

```
[In] Int[Tan[3*x]/(1 + Cos[3*x])^2,x]
```

```
[Out] -1/3*1/(1 + Cos[3*x]) - Log[Cos[3*x]]/3 + Log[1 + Cos[3*x]]/3
```

Rule 46

```
Int[((a_) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_.), x_Symbol] :> Int[E
xpandIntegrand[(a + b*x)^m*(c + d*x)^n, x], x] /; FreeQ[{a, b, c, d}, x] &&
 NeQ[b*c - a*d, 0] && ILtQ[m, 0] && IntegerQ[n] &&  !(IGtQ[n, 0] && LtQ[m +
 n + 2, 0])
```

Rule 2786

```
Int[((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m_.)*tan[(e_.) + (f_.)*(x_)]^(p
_.), x_Symbol] :> Dist[1/f, Subst[Int[x^p*((a + x)^(m - (p + 1)/2)/(a - x)^
```

```
((p + 1)/2)), x], x, b*Sin[e + f*x]], x] /; FreeQ[{a, b, e, f, m}, x] && Eq
Q[a^2 - b^2, 0] && IntegerQ[(p + 1)/2]
```

Rubi steps

$$\begin{aligned}\text{integral} &= -\left(\frac{1}{3}\text{Subst}\left(\int \frac{1}{x(1+x)^2}\,dx, x, \cos(3x)\right)\right)\\ &= -\left(\frac{1}{3}\text{Subst}\left(\int \left(\frac{1}{-1-x}+\frac{1}{x}-\frac{1}{(1+x)^2}\right)dx, x, \cos(3x)\right)\right)\\ &= -\frac{1}{3(1+\cos(3x))}-\frac{1}{3}\log(\cos(3x))+\frac{1}{3}\log(1+\cos(3x))\end{aligned}$$

Mathematica [A] (verified)

Time = 0.04 (sec) , antiderivative size = 49, normalized size of antiderivative = 1.48

$$\int \frac{\tan(3x)}{(1+\cos(3x))^2}\,dx = \frac{-2\cos^2\left(\frac{3x}{2}\right)+\cos^4\left(\frac{3x}{2}\right)\left(8\log\left(\cos\left(\frac{3x}{2}\right)\right)-4\log(\cos(3x))\right)}{3(1+\cos(3x))^2}$$

```
[In] Integrate[Tan[3*x]/(1 + Cos[3*x])^2,x]
[Out] (-2*Cos[(3*x)/2]^2 + Cos[(3*x)/2]^4*(8*Log[Cos[(3*x)/2]] - 4*Log[Cos[3*x]])
)/(3*(1 + Cos[3*x])^2)
```

Maple [A] (verified)

Time = 0.95 (sec) , antiderivative size = 28, normalized size of antiderivative = 0.85

method	result	size
derivativedivides	$-\frac{1}{3(1+\cos(3x))}-\frac{\ln(\cos(3x))}{3}+\frac{\ln(1+\cos(3x))}{3}$	28
default	$-\frac{1}{3(1+\cos(3x))}-\frac{\ln(\cos(3x))}{3}+\frac{\ln(1+\cos(3x))}{3}$	28
risch	$-\frac{2\,e^{3ix}}{3(e^{3ix}+1)^2}+\frac{2\ln(e^{3ix}+1)}{3}-\frac{\ln(e^{6ix}+1)}{3}$	38

```
[In] int(tan(3*x)/(1+cos(3*x))^2,x,method=_RETURNVERBOSE)
[Out] -1/3/(1+cos(3*x))-1/3*ln(cos(3*x))+1/3*ln(1+cos(3*x))
```

Fricas [A] (verification not implemented)

none

Time = 0.26 (sec) , antiderivative size = 43, normalized size of antiderivative = 1.30

$$\int \frac{\tan(3x)}{(1+\cos(3x))^2}\,dx$$
$$= -\frac{(\cos(3x)+1)\log(-\cos(3x)) - (\cos(3x)+1)\log\left(\frac{1}{2}\cos(3x)+\frac{1}{2}\right)+1}{3(\cos(3x)+1)}$$

```
[In] integrate(tan(3*x)/(1+cos(3*x))^2,x, algorithm="fricas")
[Out] -1/3*((cos(3*x) + 1)*log(-cos(3*x)) - (cos(3*x) + 1)*log(1/2*cos(3*x) + 1/2
) + 1)/(cos(3*x) + 1)
```

Sympy [F]

$$\int \frac{\tan(3x)}{(1+\cos(3x))^2}\,dx = \int \frac{\tan(3x)}{(\cos(3x)+1)^2}\,dx$$

```
[In] integrate(tan(3*x)/(1+cos(3*x))**2,x)
[Out] Integral(tan(3*x)/(cos(3*x) + 1)**2, x)
```

Maxima [A] (verification not implemented)

none

Time = 0.22 (sec) , antiderivative size = 27, normalized size of antiderivative = 0.82

$$\int \frac{\tan(3x)}{(1+\cos(3x))^2}\,dx = -\frac{1}{3(\cos(3x)+1)} + \frac{1}{3}\log(\cos(3x)+1) - \frac{1}{3}\log(\cos(3x))$$

```
[In] integrate(tan(3*x)/(1+cos(3*x))^2,x, algorithm="maxima")
[Out] -1/3/(cos(3*x) + 1) + 1/3*log(cos(3*x) + 1) - 1/3*log(cos(3*x))
```

Giac [A] (verification not implemented)

none

Time = 0.29 (sec) , antiderivative size = 28, normalized size of antiderivative = 0.85

$$\int \frac{\tan(3x)}{(1+\cos(3x))^2}\,dx = -\frac{1}{3\left(\cos\left(3\,x\right)+1\right)} + \frac{1}{3}\,\log\left(\cos\left(3\,x\right)+1\right) - \frac{1}{3}\,\log\left(\left|\cos\left(3\,x\right)\right|\right)$$

```
[In] integrate(tan(3*x)/(1+cos(3*x))^2,x, algorithm="giac")
[Out] -1/3/(cos(3*x) + 1) + 1/3*log(cos(3*x) + 1) - 1/3*log(abs(cos(3*x)))
```

Mupad [B] (verification not implemented)

Time = 14.14 (sec) , antiderivative size = 20, normalized size of antiderivative = 0.61

$$\int \frac{\tan(3x)}{(1+\cos(3x))^2}\,dx = -\frac{\ln\left({\tan\left(\frac{3\,x}{2}\right)}^2-1\right)}{3} - \frac{{\tan\left(\frac{3\,x}{2}\right)}^2}{6}$$

```
[In] int(tan(3*x)/(cos(3*x) + 1)^2,x)
[Out] - log(tan((3*x)/2)^2 - 1)/3 - tan((3*x)/2)^2/6
```

3.10 $\int \frac{\tan^4(x)}{a+b\cos(x)}\,dx$

Optimal result

Integrand size = 13, antiderivative size = 113

$$\int \frac{\tan^4(x)}{a+b\cos(x)}\,dx = \frac{2(a-b)^{3/2}(a+b)^{3/2}\arctan\left(\frac{\sqrt{a-b}\tan\left(\frac{x}{2}\right)}{\sqrt{a+b}}\right)}{a^4} + \frac{b(3a^2-2b^2)\,\mathrm{arctanh}(\sin(x))}{2a^4} - \frac{(4a^2-3b^2)\tan(x)}{3a^3} - \frac{b\sec(x)\tan(x)}{2a^2} + \frac{\sec^2(x)\tan(x)}{3a}$$

```
[Out] 2*(a-b)^(3/2)*(a+b)^(3/2)*arctan((a-b)^(1/2)*tan(1/2*x)/(a+b)^(1/2))/a^4+1/
2*b*(3*a^2-2*b^2)*arctanh(sin(x))/a^4-1/3*(4*a^2-3*b^2)*tan(x)/a^3-1/2*b*se
c(x)*tan(x)/a^2+1/3*sec(x)^2*tan(x)/a
```

Rubi [A] (verified)

Time = 0.64 (sec) , antiderivative size = 113, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 6, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.462, Rules used = {2804, 3134, 3080, 3855, 2738, 211}

$$\int \frac{\tan^4(x)}{a+b\cos(x)}\,dx = \frac{2(a-b)^{3/2}(a+b)^{3/2}\arctan\left(\frac{\sqrt{a-b}\tan\left(\frac{x}{2}\right)}{\sqrt{a+b}}\right)}{a^4} - \frac{b\tan(x)\sec(x)}{2a^2} + \frac{b(3a^2-2b^2)\,\mathrm{arctanh}(\sin(x))}{2a^4} - \frac{(4a^2-3b^2)\tan(x)}{3a^3} + \frac{\tan(x)\sec^2(x)}{3a}$$

```
[In] Int[Tan[x]^4/(a + b*Cos[x]),x]
```

```
[Out] (2*(a - b)^(3/2)*(a + b)^(3/2)*ArcTan[(Sqrt[a - b]*Tan[x/2])/Sqrt[a + b]])/
a^4 + (b*(3*a^2 - 2*b^2)*ArcTanh[Sin[x]])/(2*a^4) - ((4*a^2 - 3*b^2)*Tan[x]
)/(3*a^3) - (b*Sec[x]*Tan[x])/(2*a^2) + (Sec[x]^2*Tan[x])/(3*a)
```

Rule 211

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

Rule 2738

```
Int[((a_) + (b_.)*sin[Pi/2 + (c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{
e = FreeFactors[Tan[(c + d*x)/2], x]}, Dist[2*(e/d), Subst[Int[1/(a + b + (
a - b)*e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x]
&& NeQ[a^2 - b^2, 0]
```

Rule 2804

```
Int[((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m_)/tan[(e_.) + (f_.)*(x_)]^4,
x_Symbol] :> Simp[(-Cos[e + f*x])*((a + b*Sin[e + f*x])^(m + 1)/(3*a*f*Sin[
e + f*x]^3)), x] + (-Dist[1/(6*a^2), Int[((a + b*Sin[e + f*x])^m/Sin[e + f*
x]^2)*Simp[8*a^2 - b^2*(m - 1)*(m - 2) + a*b*m*Sin[e + f*x] - (6*a^2 - b^2*
m*(m - 2))*Sin[e + f*x]^2, x], x], x] - Simp[b*(m - 2)*Cos[e + f*x]*((a + b
*Sin[e + f*x])^(m + 1)/(6*a^2*f*Sin[e + f*x]^2)), x]) /; FreeQ[{a, b, e, f,
 m}, x] && NeQ[a^2 - b^2, 0] &&  !LtQ[m, -1] && IntegerQ[2*m]
```

Rule 3080

```
Int[((A_.) + (B_.)*sin[(e_.) + (f_.)*(x_)])/(((a_.) + (b_.)*sin[(e_.) + (f_
.)*(x_)])*((c_.) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Dist[(A*b
- a*B)/(b*c - a*d), Int[1/(a + b*Sin[e + f*x]), x], x] + Dist[(B*c - A*d)/(
b*c - a*d), Int[1/(c + d*Sin[e + f*x]), x], x] /; FreeQ[{a, b, c, d, e, f,
A, B}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]
```

Rule 3134

```
Int[((a_.) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m_)*((c_.) + (d_.)*sin[(e_.) +
 (f_.)*(x_)])^(n_)*((A_.) + (B_.)*sin[(e_.) + (f_.)*(x_)] + (C_.)*sin[(e_.)
 + (f_.)*(x_)]^2), x_Symbol] :> Simp[(-(A*b^2 - a*b*B + a^2*C))*Cos[e + f*x
]*(a + b*Sin[e + f*x])^(m + 1)*((c + d*Sin[e + f*x])^(n + 1)/(f*(m + 1)*(b*
c - a*d)*(a^2 - b^2))), x] + Dist[1/((m + 1)*(b*c - a*d)*(a^2 - b^2)), Int[
(a + b*Sin[e + f*x])^(m + 1)*(c + d*Sin[e + f*x])^n*Simp[(m + 1)*(b*c - a*d
)*(a*A - b*B + a*C) + d*(A*b^2 - a*b*B + a^2*C)*(m + n + 2) - (c*(A*b^2 - a
*b*B + a^2*C) + (m + 1)*(b*c - a*d)*(A*b - a*B + b*C))*Sin[e + f*x] - d*(A*
b^2 - a*b*B + a^2*C)*(m + n + 3)*Sin[e + f*x]^2, x], x], x] /; FreeQ[{a, b,
 c, d, e, f, A, B, C, n}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && N
eQ[c^2 - d^2, 0] && LtQ[m, -1] && ((EqQ[a, 0] && IntegerQ[m] &&  !IntegerQ[
n]) ||  !(IntegerQ[2*n] && LtQ[n, -1] && ((IntegerQ[n] &&  !IntegerQ[m]) ||
 EqQ[a, 0])))
```

Rule 3855

```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] :> Simp[-ArcTanh[Cos[c + d*x]]/d, x]
 /; FreeQ[{c, d}, x]
```

Rubi steps

$$\begin{aligned}
\text{integral} &= -\frac{b\sec(x)\tan(x)}{2a^2}+\frac{\sec^2(x)\tan(x)}{3a}-\frac{\int\frac{(2(4a^2-3b^2)-ab\cos(x)-3(2a^2-b^2)\cos^2(x))\sec^2(x)}{a+b\cos(x)}\,dx}{6a^2}\\
&= -\frac{(4a^2-3b^2)\tan(x)}{3a^3}-\frac{b\sec(x)\tan(x)}{2a^2}+\frac{\sec^2(x)\tan(x)}{3a}\\
&\quad-\frac{\int\frac{(-3b(3a^2-2b^2)-3a(2a^2-b^2)\cos(x))\sec(x)}{a+b\cos(x)}\,dx}{6a^3}\\
&= -\frac{(4a^2-3b^2)\tan(x)}{3a^3}-\frac{b\sec(x)\tan(x)}{2a^2}+\frac{\sec^2(x)\tan(x)}{3a}\\
&\quad+\frac{(b(3a^2-2b^2))\int\sec(x)\,dx}{2a^4}+\frac{(a^2-b^2)^2\int\frac{1}{a+b\cos(x)}\,dx}{a^4}\\
&= \frac{b(3a^2-2b^2)\operatorname{arctanh}(\sin(x))}{2a^4}-\frac{(4a^2-3b^2)\tan(x)}{3a^3}-\frac{b\sec(x)\tan(x)}{2a^2}\\
&\quad+\frac{\sec^2(x)\tan(x)}{3a}+\frac{\left(2(a^2-b^2)^2\right)\operatorname{Subst}\left(\int\frac{1}{a+b+(a-b)x^2}\,dx,x,\tan\left(\frac{x}{2}\right)\right)}{a^4}\\
&= \frac{2(a-b)^{3/2}(a+b)^{3/2}\arctan\left(\frac{\sqrt{a-b}\tan\left(\frac{x}{2}\right)}{\sqrt{a+b}}\right)}{a^4}+\frac{b(3a^2-2b^2)\operatorname{arctanh}(\sin(x))}{2a^4}\\
&\quad-\frac{(4a^2-3b^2)\tan(x)}{3a^3}-\frac{b\sec(x)\tan(x)}{2a^2}+\frac{\sec^2(x)\tan(x)}{3a}
\end{aligned}$$

Mathematica [A] (verified)

Time = 0.81 (sec) , antiderivative size = 190, normalized size of antiderivative = 1.68

$$\int\frac{\tan^4(x)}{a+b\cos(x)}\,dx=$$

$$-\frac{48(-a^2+b^2)^{3/2}\operatorname{arctanh}\left(\frac{(a-b)\tan\left(\frac{x}{2}\right)}{\sqrt{-a^2+b^2}}\right)+\sec^3(x)\left(9b(3a^2-2b^2)\cos(x)\left(\log\left(\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right)-\log\left(\mathrm{co}\right.\right.\right.}{}$$

```
[In] Integrate[Tan[x]^4/(a + b*Cos[x]),x]
```

```
[Out] -1/24*(48*(-a^2 + b^2)^(3/2)*ArcTanh[((a - b)*Tan[x/2])/Sqrt[-a^2 + b^2]] +
 Sec[x]^3*(9*b*(3*a^2 - 2*b^2)*Cos[x]*(Log[Cos[x/2] - Sin[x/2]] - Log[Cos[x
/2] + Sin[x/2]]) + 3*b*(3*a^2 - 2*b^2)*Cos[3*x]*(Log[Cos[x/2] - Sin[x/2]] -
 Log[Cos[x/2] + Sin[x/2]]) + 2*a*(-3*b^2*Sin[x] + 3*a*b*Sin[2*x] + (4*a^2 -
 3*b^2)*Sin[3*x])))/a^4
```

Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 218 vs. $2(95) = 190$.

Time = 1.10 (sec) , antiderivative size = 219, normalized size of antiderivative = 1.94

method	result
default	$-\frac{1}{3a\left(\tan\left(\frac{x}{2}\right)+1\right)^{3}}-\frac{-a-b}{2a^{2}\left(\tan\left(\frac{x}{2}\right)+1\right)^{2}}-\frac{-2a^{2}+ab+2b^{2}}{2a^{3}\left(\tan\left(\frac{x}{2}\right)+1\right)}+\frac{b\left(3a^{2}-2b^{2}\right)\ln\left(\tan\left(\frac{x}{2}\right)+1\right)}{2a^{4}}+\frac{2\left(a^{4}-2a^{2}b^{2}+b^{4}\right)\arctan\left(\frac{(a-b)\tan(}{\sqrt{(a-b)(a-}}\right.}{a^{4}\sqrt{(a-b)(a+b)}}$
risch	$\frac{i\left(3ab\,e^{5ix}-12a^{2}e^{4ix}+6b^{2}e^{4ix}-12\,e^{2ix}a^{2}+12\,e^{2ix}b^{2}-3b\,e^{ix}a-8a^{2}+6b^{2}\right)}{3a^{3}\left(e^{2ix}+1\right)^{3}}-\frac{3b\ln\left(e^{ix}-i\right)}{2a^{2}}+\frac{b^{3}\ln\left(e^{ix}-i\right)}{a^{4}}+\frac{\sqrt{-a^{2}+b^{2}}\,\ln\left(e^{ix}-\frac{i\sqrt{}}{}\right.}{a^{2}}$

```
[In] int(tan(x)^4/(a+cos(x)*b),x,method=_RETURNVERBOSE)
```

```
[Out] -1/3/a/(tan(1/2*x)+1)^3-1/2*(-a-b)/a^2/(tan(1/2*x)+1)^2-1/2*(-2*a^2+a*b+2*b
^2)/a^3/(tan(1/2*x)+1)+1/2*b*(3*a^2-2*b^2)/a^4*ln(tan(1/2*x)+1)+2*(a^4-2*a^
2*b^2+b^4)/a^4/((a-b)*(a+b))^(1/2)*arctan((a-b)*tan(1/2*x)/((a-b)*(a+b))^(1
/2))-1/3/a/(tan(1/2*x)-1)^3-1/2*(a+b)/a^2/(tan(1/2*x)-1)^2-1/2*(-2*a^2+a*b+
2*b^2)/a^3/(tan(1/2*x)-1)-1/2*b*(3*a^2-2*b^2)/a^4*ln(tan(1/2*x)-1)
```

Fricas [A] (verification not implemented)

none

Time = 0.38 (sec) , antiderivative size = 332, normalized size of antiderivative = 2.94

$$\int \frac{\tan^4(x)}{a+b\cos(x)}\,dx = \left[-\frac{6\left(a^2-b^2\right)\sqrt{-a^2+b^2}\cos\left(x\right)^3\log\left(\frac{2\,ab\cos(x)+\left(2\,a^2-b^2\right)\cos(x)^2+2\,\sqrt{-a^2+b^2}(a\cos(x)+b)\sin(x)-a^2+2\,b^2}{b^2\cos(x)^2+2\,ab\cos(x)+a^2}\right)-3\left(3\,a^2b-\right.}{}\right.$$

```
[In] integrate(tan(x)^4/(a+b*cos(x)),x, algorithm="fricas")
```

```
[Out] [-1/12*(6*(a^2 - b^2)*sqrt(-a^2 + b^2)*cos(x)^3*log((2*a*b*cos(x) + (2*a^2
- b^2)*cos(x)^2 + 2*sqrt(-a^2 + b^2)*(a*cos(x) + b)*sin(x) - a^2 + 2*b^2)/(
b^2*cos(x)^2 + 2*a*b*cos(x) + a^2)) - 3*(3*a^2*b - 2*b^3)*cos(x)^3*log(sin(
x) + 1) + 3*(3*a^2*b - 2*b^3)*cos(x)^3*log(-sin(x) + 1) + 2*(3*a^2*b*cos(x)
 - 2*a^3 + 2*(4*a^3 - 3*a*b^2)*cos(x)^2)*sin(x))/(a^4*cos(x)^3), 1/12*(12*(
a^2 - b^2)^(3/2)*arctan(-(a*cos(x) + b)/(sqrt(a^2 - b^2)*sin(x)))*cos(x)^3
+ 3*(3*a^2*b - 2*b^3)*cos(x)^3*log(sin(x) + 1) - 3*(3*a^2*b - 2*b^3)*cos(x)
^3*log(-sin(x) + 1) - 2*(3*a^2*b*cos(x) - 2*a^3 + 2*(4*a^3 - 3*a*b^2)*cos(x
)^2)*sin(x))/(a^4*cos(x)^3)]
```

Sympy [F]

$$\int \frac{\tan^4(x)}{a+b\cos(x)}\,dx = \int \frac{\tan^4{(x)}}{a+b\cos{(x)}}\,dx$$

```
[In] integrate(tan(x)**4/(a+b*cos(x)),x)
```

```
[Out] Integral(tan(x)**4/(a + b*cos(x)), x)
```

Maxima [F(-2)]

Exception generated.

$$\int \frac{\tan^4(x)}{a+b\cos(x)}\,dx = \text{Exception raised: ValueError}$$

```
[In] integrate(tan(x)^4/(a+b*cos(x)),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*b^2-4*a^2>0)', see `assume?` for
more de
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 226 vs. 2(95) = 190.

Time = 0.31 (sec) , antiderivative size = 226, normalized size of antiderivative = 2.00

$$\begin{aligned}\int \frac{\tan^4(x)}{a+b\cos(x)}\,dx &= \frac{(3\,a^2b-2\,b^3)\log\left(\left|\tan\left(\frac{1}{2}\,x\right)+1\right|\right)}{2\,a^4} - \frac{(3\,a^2b-2\,b^3)\log\left(\left|\tan\left(\frac{1}{2}\,x\right)-1\right|\right)}{2\,a^4} \\ &- \frac{2\,(a^4-2\,a^2b^2+b^4)\left(\pi\left\lfloor\frac{x}{2\pi}+\frac{1}{2}\right\rfloor\operatorname{sgn}(-2\,a+2\,b)+\arctan\left(-\frac{a\tan\left(\frac{1}{2}\,x\right)-b\tan\left(\frac{1}{2}\,x\right)}{\sqrt{a^2-b^2}}\right)\right)}{\sqrt{a^2-b^2}a^4} \\ &+ \frac{6\,a^2\tan\left(\frac{1}{2}\,x\right)^5-3\,ab\tan\left(\frac{1}{2}\,x\right)^5-6\,b^2\tan\left(\frac{1}{2}\,x\right)^5-20\,a^2\tan\left(\frac{1}{2}\,x\right)^3+12\,b^2\tan\left(\frac{1}{2}\,x\right)^3+6\,a^2\tan\left(\frac{1}{2}\,x\right)+}{3\left(\tan\left(\frac{1}{2}\,x\right)^2-1\right)^3a^3}\end{aligned}$$

```
[In] integrate(tan(x)^4/(a+b*cos(x)),x, algorithm="giac")
```

```
[Out] 1/2*(3*a^2*b - 2*b^3)*log(abs(tan(1/2*x) + 1))/a^4 - 1/2*(3*a^2*b - 2*b^3)*
log(abs(tan(1/2*x) - 1))/a^4 - 2*(a^4 - 2*a^2*b^2 + b^4)*(pi*floor(1/2*x/pi
 + 1/2)*sgn(-2*a + 2*b) + arctan(-(a*tan(1/2*x) - b*tan(1/2*x))/sqrt(a^2 -
b^2)))/(sqrt(a^2 - b^2)*a^4) + 1/3*(6*a^2*tan(1/2*x)^5 - 3*a*b*tan(1/2*x)^5
 - 6*b^2*tan(1/2*x)^5 - 20*a^2*tan(1/2*x)^3 + 12*b^2*tan(1/2*x)^3 + 6*a^2*t
an(1/2*x) + 3*a*b*tan(1/2*x) - 6*b^2*tan(1/2*x))/((tan(1/2*x)^2 - 1)^3*a^3)
```

Mupad [B] (verification not implemented)

Time = 14.85 (sec) , antiderivative size = 1666, normalized size of antiderivative = 14.74

$$\int \frac{\tan^4(x)}{a + b\cos(x)}\, dx = \text{Too large to display}$$

[In] int(tan(x)^4/(a + b*cos(x)),x)

[Out] (2*atanh((64*tan(x/2)*(b^6 - a^6 - 3*a^2*b^4 + 3*a^4*b^2)^(1/2))/(112*a*b^2 + 128*a^2*b - 64*a^3 - 352*b^3 + (16*b^4)/a + (320*b^5)/a^2 - (112*b^6)/a^3 - (96*b^7)/a^4 + (48*b^8)/a^5) + (144*b^2*tan(x/2)*(b^6 - a^6 - 3*a^2*b^4 + 3*a^4*b^2)^(1/2))/(16*a*b^4 + 128*a^4*b - 64*a^5 + 320*b^5 - 352*a^2*b^3 + 112*a^3*b^2 - (112*b^6)/a - (96*b^7)/a^2 + (48*b^8)/a^3) + (80*b^3*tan(x/2)*(b^6 - a^6 - 3*a^2*b^4 + 3*a^4*b^2)^(1/2))/(320*a*b^5 + 128*a^5*b - 64*a^6 - 112*b^6 + 16*a^2*b^4 - 352*a^3*b^3 + 112*a^4*b^2 - (96*b^7)/a + (48*b^8)/a^2) - (144*b^4*tan(x/2)*(b^6 - a^6 - 3*a^2*b^4 + 3*a^4*b^2)^(1/2))/(128*a^6*b - 112*a*b^6 - 64*a^7 - 96*b^7 + 320*a^2*b^5 + 16*a^3*b^4 - 352*a^4*b^3 + 112*a^5*b^2 + (48*b^8)/a) + (48*b^5*tan(x/2)*(b^6 - a^6 - 3*a^2*b^4 + 3*a^4*b^2)^(1/2))/(128*a^7*b - 96*a*b^7 - 64*a^8 + 48*b^8 - 112*a^2*b^6 + 320*a^3*b^5 + 16*a^4*b^4 - 352*a^5*b^3 + 112*a^6*b^2) - (192*b*tan(x/2)*(b^6 - a^6 - 3*a^2*b^4 + 3*a^4*b^2)^(1/2))/(128*a^3*b - 352*a*b^3 - 64*a^4 + 16*b^4 + 112*a^2*b^2 + (320*b^5)/a - (112*b^6)/a^2 - (96*b^7)/a^3 + (48*b^8)/a^4))*(-(a + b)^3*(a - b)^3)^(1/2))/a^4 - ((4*tan(x/2)^3*(5*a^2 - 3*b^2))/(3*a^3) - (tan(x/2)*(a*b + 2*a^2 - 2*b^2))/a^3 + (tan(x/2)^5*(a*b - 2*a^2 + 2*b^2))/a^3)/(3*tan(x/2)^2 - 3*tan(x/2)^4 + tan(x/2)^6 - 1) - (atan(((((3*a^2*b)/2 - b^3)*((((3*a^2*b)/2 - b^3)*((8*(2*a^12*b - 4*a^13 + 4*a^8*b^5 - 6*a^9*b^4 - 6*a^10*b^3 + 10*a^11*b^2))/a^9 - (8*tan(x/2)*((3*a^2*b)/2 - b^3)*(8*a^10*b + 8*a^8*b^3 - 16*a^9*b^2))/a^10))/a^4 - (8*tan(x/2)*(16*a*b^8 - 4*a^8*b + 4*a^9 - 8*b^9 + 16*a^2*b^7 - 48*a^3*b^6 + 3*a^4*b^5 + 39*a^5*b^4 - 11*a^6*b^3 - 7*a^7*b^2))/a^6)*1i)/a^4 - (((3*a^2*b)/2 - b^3)*((((3*a^2*b)/2 - b^3)*((8*(2*a^12*b - 4*a^13 + 4*a^8*b^5 - 6*a^9*b^4 - 6*a^10*b^3 + 10*a^11*b^2))/a^9 + (8*tan(x/2)*((3*a^2*b)/2 - b^3)*(8*a^10*b + 8*a^8*b^3 - 16*a^9*b^2))/a^10))/a^4 + (8*tan(x/2)*(16*a*b^8 - 4*a^8*b + 4*a^9 - 8*b^9 + 16*a^2*b^7 - 48*a^3*b^6 + 3*a^4*b^5 + 39*a^5*b^4 - 11*a^6*b^3 - 7*a^7*b^2))/a^6)*1i)/a^4)/((16*(6*a*b^10 - 6*a^10*b - 4*b^11 + 18*a^2*b^9 - 31*a^3*b^8 - 26*a^4*b^7 + 59*a^5*b^6 + 8*a^6*b^5 - 49*a^7*b^4 + 10*a^8*b^3 + 15*a^9*b^2))/a^9 + (((3*a^2*b)/2 - b^3)*((((3*a^2*b)/2 - b^3)*((8*(2*a^12*b - 4*a^13 + 4*a^8*b^5 - 6*a^9*b^4 - 6*a^10*b^3 + 10*a^11*b^2))/a^9 - (8*tan(x/2)*((3*a^2*b)/2 - b^3)*(8*a^10*b + 8*a^8*b^3 - 16*a^9*b^2))/a^10))/a^4 - (8*tan(x/2)*(16*a*b^8 - 4*a^8*b + 4*a^9 - 8*b^9 + 16*a^2*b^7 - 48*a^3*b^6 + 3*a^4*b^5 + 39*a^5*b^4 - 11*a^6*b^3 - 7*a^7*b^2))/a^6))/a^4 + (((3*a^2*b)/2 - b^3)*((((3*a^2*b)/2 - b^3)*((8*(2*a^12*b - 4*a^13 + 4*a^8*b^5 - 6*a^9*b^4 - 6*a^10*b^3 + 10*a^11*b^2))/a^9 + (8*tan(x/2)*((3*a^2*b)/2 - b^3)*(8*a^10*b + 8*a^8*b^3 - 16*a^9*b^2))/a^10))/a^4 + (8*tan(x/2)*(16*a*b^8 - 4*a^8*b + 4*a

```
^9 - 8*b^9 + 16*a^2*b^7 - 48*a^3*b^6 + 3*a^4*b^5 + 39*a^5*b^4 - 11*a^6*b^3
- 7*a^7*b^2))/a^6))/a^4))*((3*a^2*b)/2 - b^3)*2i)/a^4
```

3.11 $\int \frac{\tan^3(x)}{a+b\cos(x)}\,dx$

Optimal result

Integrand size = 13, antiderivative size = 57

$$\int \frac{\tan^3(x)}{a+b\cos(x)}\,dx=\frac{(a^2-b^2)\log(\cos(x))}{a^3}-\frac{(a^2-b^2)\log(a+b\cos(x))}{a^3}-\frac{b\sec(x)}{a^2}+\frac{\sec^2(x)}{2a}$$

```
[Out] (a^2-b^2)*ln(cos(x))/a^3-(a^2-b^2)*ln(a+b*cos(x))/a^3-b*sec(x)/a^2+1/2*sec(
x)^2/a
```

Rubi [A] (verified)

Time = 0.12 (sec) , antiderivative size = 57, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.154, Rules used = {2800, 908}

$$\int \frac{\tan^3(x)}{a+b\cos(x)}\,dx=-\frac{b\sec(x)}{a^2}+\frac{(a^2-b^2)\log(\cos(x))}{a^3}-\frac{(a^2-b^2)\log(a+b\cos(x))}{a^3}+\frac{\sec^2(x)}{2a}$$

```
[In] Int[Tan[x]^3/(a + b*Cos[x]),x]
```

```
[Out] ((a^2 - b^2)*Log[Cos[x]])/a^3 - ((a^2 - b^2)*Log[a + b*Cos[x]])/a^3 - (b*Se
c[x])/a^2 + Sec[x]^2/(2*a)
```

Rule 908

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))^(n_)*((a_) + (c_.)*(x_)^
2)^(p_.), x_Symbol] :> Int[ExpandIntegrand[(d + e*x)^m*(f + g*x)^n*(a + c*x
^2)^p, x], x] /; FreeQ[{a, c, d, e, f, g}, x] && NeQ[e*f - d*g, 0] && NeQ[c
*d^2 + a*e^2, 0] && IntegerQ[p] && ((EqQ[p, 1] && IntegersQ[m, n]) || (ILtQ
[m, 0] && ILtQ[n, 0]))
```

Rule 2800

```
Int[((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m_.)*tan[(e_.) + (f_.)*(x_)]^(p
_.), x_Symbol] :> Dist[1/f, Subst[Int[(x^p*(a + x)^m)/(b^2 - x^2)^((p + 1)/
2), x], x, b*Sin[e + f*x]], x] /; FreeQ[{a, b, e, f, m}, x] && NeQ[a^2 - b^
2, 0] && IntegerQ[(p + 1)/2]
```

Rubi steps

$$\begin{aligned}\text{integral} &= -\text{Subst}\left(\int \frac{b^2-x^2}{x^3(a+x)}\,dx, x, b\cos(x)\right)\\ &= -\text{Subst}\left(\int \left(\frac{b^2}{ax^3}-\frac{b^2}{a^2x^2}+\frac{-a^2+b^2}{a^3x}+\frac{a^2-b^2}{a^3(a+x)}\right)dx, x, b\cos(x)\right)\\ &= \frac{(a^2-b^2)\log(\cos(x))}{a^3}-\frac{(a^2-b^2)\log(a+b\cos(x))}{a^3}-\frac{b\sec(x)}{a^2}+\frac{\sec^2(x)}{2a}\end{aligned}$$

Mathematica [A] (verified)

Time = 0.08 (sec) , antiderivative size = 46, normalized size of antiderivative = 0.81

$$\int \frac{\tan^3(x)}{a+b\cos(x)}\,dx = \frac{2(a^2-b^2)(\log(\cos(x))-\log(a+b\cos(x)))-2ab\sec(x)+a^2\sec^2(x)}{2a^3}$$

```
[In] Integrate[Tan[x]^3/(a + b*Cos[x]),x]
[Out] (2*(a^2 - b^2)*(Log[Cos[x]] - Log[a + b*Cos[x]]) - 2*a*b*Sec[x] + a^2*Sec[x
]^2)/(2*a^3)
```

Maple [A] (verified)

Time = 0.66 (sec) , antiderivative size = 58, normalized size of antiderivative = 1.02

method	result	size
default	$-\frac{b}{a^2\cos(x)}+\frac{(a^2-b^2)\ln(\cos(x))}{a^3}+\frac{1}{2a\cos(x)^2}-\frac{(a^2-b^2)\ln(a+\cos(x)b)}{a^3}$	58
risch	$-\frac{2(e^{3ix}b-a\,e^{2ix}+e^{ix}b)}{(e^{2ix}+1)^2a^2}-\frac{\ln\left(e^{2ix}+\frac{2a\,e^{ix}}{b}+1\right)}{a}+\frac{\ln\left(e^{2ix}+\frac{2a\,e^{ix}}{b}+1\right)b^2}{a^3}+\frac{\ln(e^{2ix}+1)}{a}-\frac{\ln(e^{2ix}+1)b^2}{a^3}$	117

```
[In] int(tan(x)^3/(a+cos(x)*b),x,method=_RETURNVERBOSE)
[Out] -1/a^2*b/cos(x)+(a^2-b^2)*ln(cos(x))/a^3+1/2/a/cos(x)^2-(a^2-b^2)*ln(a+cos(
x)*b)/a^3
```

Fricas [A] (verification not implemented)

none

Time = 0.29 (sec) , antiderivative size = 66, normalized size of antiderivative = 1.16

$$\int \frac{\tan^3(x)}{a+b\cos(x)}\,dx = -\frac{2\left(a^2-b^2\right)\cos\left(x\right)^2\log\left(-b\cos\left(x\right)-a\right)-2\left(a^2-b^2\right)\cos\left(x\right)^2\log\left(-\cos\left(x\right)\right)+2\,ab\cos\left(x\right)-a^2}{2\,a^3\cos\left(x\right)^2}$$

```
[In] integrate(tan(x)^3/(a+b*cos(x)),x, algorithm="fricas")
[Out] -1/2*(2*(a^2 - b^2)*cos(x)^2*log(-b*cos(x) - a) - 2*(a^2 - b^2)*cos(x)^2*lo
g(-cos(x)) + 2*a*b*cos(x) - a^2)/(a^3*cos(x)^2)
```

Sympy [F]

$$\int \frac{\tan^3(x)}{a+b\cos(x)}\,dx = \int \frac{\tan^3\left(x\right)}{a+b\cos\left(x\right)}\,dx$$

```
[In] integrate(tan(x)**3/(a+b*cos(x)),x)
[Out] Integral(tan(x)**3/(a + b*cos(x)), x)
```

Maxima [A] (verification not implemented)

none

Time = 0.28 (sec) , antiderivative size = 56, normalized size of antiderivative = 0.98

$$\int \frac{\tan^3(x)}{a+b\cos(x)}\,dx = -\frac{\left(a^2-b^2\right)\log\left(b\cos\left(x\right)+a\right)}{a^3}+\frac{\left(a^2-b^2\right)\log\left(\cos\left(x\right)\right)}{a^3}-\frac{2\,b\cos\left(x\right)-a}{2\,a^2\cos\left(x\right)^2}$$

```
[In] integrate(tan(x)^3/(a+b*cos(x)),x, algorithm="maxima")
[Out] -(a^2 - b^2)*log(b*cos(x) + a)/a^3 + (a^2 - b^2)*log(cos(x))/a^3 - 1/2*(2*b
*cos(x) - a)/(a^2*cos(x)^2)
```

Giac [A] (verification not implemented)

none

Time = 0.33 (sec) , antiderivative size = 66, normalized size of antiderivative = 1.16

$$\int \frac{\tan^3(x)}{a+b\cos(x)}\,dx = \frac{(a^2-b^2)\log(|\cos(x)|)}{a^3} - \frac{(a^2b-b^3)\log(|b\cos(x)+a|)}{a^3b} - \frac{2\,ab\cos(x)-a^2}{2\,a^3\cos(x)^2}$$

```
[In] integrate(tan(x)^3/(a+b*cos(x)),x, algorithm="giac")
```

```
[Out] (a^2 - b^2)*log(abs(cos(x)))/a^3 - (a^2*b - b^3)*log(abs(b*cos(x) + a))/(a^
3*b) - 1/2*(2*a*b*cos(x) - a^2)/(a^3*cos(x)^2)
```

Mupad [B] (verification not implemented)

Time = 14.54 (sec) , antiderivative size = 115, normalized size of antiderivative = 2.02

$$\int \frac{\tan^3(x)}{a+b\cos(x)}\,dx = -\frac{2\,a^2\,\mathrm{atanh}\left(\frac{a\tan\left(\frac{x}{2}\right)^2}{-b\tan\left(\frac{x}{2}\right)^2+a+b}\right) - 2\,b^2\,\mathrm{atanh}\left(\frac{a\tan\left(\frac{x}{2}\right)^2}{-b\tan\left(\frac{x}{2}\right)^2+a+b}\right)}{a^3} - \frac{2\,a\,b - \tan\left(\frac{x}{2}\right)^2\left(2\,a^2+2\,b\,a\right)}{a^3\tan\left(\frac{x}{2}\right)^4 - 2\,a^3\tan\left(\frac{x}{2}\right)^2 + a^3}$$

```
[In] int(tan(x)^3/(a + b*cos(x)),x)
```

```
[Out] - (2*a^2*atanh((a*tan(x/2)^2)/(a + b - b*tan(x/2)^2)) - 2*b^2*atanh((a*tan(
x/2)^2)/(a + b - b*tan(x/2)^2)))/a^3 - (2*a*b - tan(x/2)^2*(2*a*b + 2*a^2))
/(a^3 - 2*a^3*tan(x/2)^2 + a^3*tan(x/2)^4)
```

3.12 $\int \frac{\tan^2(x)}{a+b\cos(x)}\,dx$

Optimal result

Integrand size = 13, antiderivative size = 61

$$\int \frac{\tan^2(x)}{a+b\cos(x)}\,dx = -\frac{2\sqrt{a-b}\sqrt{a+b}\,\text{arctan}\left(\frac{\sqrt{a-b}\tan\left(\frac{x}{2}\right)}{\sqrt{a+b}}\right)}{a^2} - \frac{b\,\text{arctanh}(\sin(x))}{a^2} + \frac{\tan(x)}{a}$$

```
[Out] -b*arctanh(sin(x))/a^2-2*arctan((a-b)^(1/2)*tan(1/2*x)/(a+b)^(1/2))*(a-b)^(
1/2)*(a+b)^(1/2)/a^2+tan(x)/a
```

Rubi [A] (verified)

Time = 0.35 (sec) , antiderivative size = 61, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 6, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.462, Rules used = {2802, 3135, 3080, 3855, 2738, 211}

$$\int \frac{\tan^2(x)}{a+b\cos(x)}\,dx = -\frac{2\sqrt{a-b}\sqrt{a+b}\,\text{arctan}\left(\frac{\sqrt{a-b}\tan\left(\frac{x}{2}\right)}{\sqrt{a+b}}\right)}{a^2} - \frac{b\,\text{arctanh}(\sin(x))}{a^2} + \frac{\tan(x)}{a}$$

```
[In] Int[Tan[x]^2/(a + b*Cos[x]),x]
```

```
[Out] (-2*Sqrt[a - b]*Sqrt[a + b]*ArcTan[(Sqrt[a - b]*Tan[x/2])/Sqrt[a + b]])/a^2
 - (b*ArcTanh[Sin[x]])/a^2 + Tan[x]/a
```

Rule 211

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

Rule 2738

```
Int[((a_) + (b_.)*sin[Pi/2 + (c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{
e = FreeFactors[Tan[(c + d*x)/2], x]}, Dist[2*(e/d), Subst[Int[1/(a + b + (
a - b)*e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x]
&& NeQ[a^2 - b^2, 0]
```

Rule 2802

```
Int[((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m_)/tan[(e_.) + (f_.)*(x_)]^2,
x_Symbol] :> Int[(a + b*Sin[e + f*x])^m*((1 - Sin[e + f*x]^2)/Sin[e + f*x]^
2), x] /; FreeQ[{a, b, e, f, m}, x] && NeQ[a^2 - b^2, 0]
```

Rule 3080

```
Int[((A_.) + (B_.)*sin[(e_.) + (f_.)*(x_)])/(((a_.) + (b_.)*sin[(e_.) + (f_
.)*(x_)])*((c_.) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Dist[(A*b
- a*B)/(b*c - a*d), Int[1/(a + b*Sin[e + f*x]), x], x] + Dist[(B*c - A*d)/(
b*c - a*d), Int[1/(c + d*Sin[e + f*x]), x], x] /; FreeQ[{a, b, c, d, e, f,
A, B}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]
```

Rule 3135

```
Int[((a_.) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m_)*((c_.) + (d_.)*sin[(e_.) +
 (f_.)*(x_)])^(n_)*((A_.) + (C_.)*sin[(e_.) + (f_.)*(x_)]^2), x_Symbol] :>
Simp[(-(A*b^2 + a^2*C))*Cos[e + f*x]*(a + b*Sin[e + f*x])^(m + 1)*((c + d*S
in[e + f*x])^(n + 1)/(f*(m + 1)*(b*c - a*d)*(a^2 - b^2))), x] + Dist[1/((m
+ 1)*(b*c - a*d)*(a^2 - b^2)), Int[(a + b*Sin[e + f*x])^(m + 1)*(c + d*Sin[
e + f*x])^n*Simp[a*(m + 1)*(b*c - a*d)*(A + C) + d*(A*b^2 + a^2*C)*(m + n +
 2) - (c*(A*b^2 + a^2*C) + b*(m + 1)*(b*c - a*d)*(A + C))*Sin[e + f*x] - d*
(A*b^2 + a^2*C)*(m + n + 3)*Sin[e + f*x]^2, x], x], x] /; FreeQ[{a, b, c, d
, e, f, A, C, n}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 -
 d^2, 0] && LtQ[m, -1] && ((EqQ[a, 0] && IntegerQ[m] &&  !IntegerQ[n]) ||
!(IntegerQ[2*n] && LtQ[n, -1] && ((IntegerQ[n] &&  !IntegerQ[m]) || EqQ[a,
0])))
```

Rule 3855

```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] :> Simp[-ArcTanh[Cos[c + d*x]]/d, x]
 /; FreeQ[{c, d}, x]
```

Rubi steps

$$\text{integral} = \int \frac{(1-\cos^2(x))\sec^2(x)}{a+b\cos(x)}\,dx$$

$$= \frac{\tan(x)}{a} + \frac{\int \frac{(-b-a\cos(x))\sec(x)}{a+b\cos(x)}\,dx}{a}$$

$$= \frac{\tan(x)}{a} - \frac{b\int \sec(x)\,dx}{a^2} + \frac{(-a^2+b^2)\int \frac{1}{a+b\cos(x)}\,dx}{a^2}$$

$$= -\frac{b\text{arctanh}(\sin(x))}{a^2} + \frac{\tan(x)}{a} + \frac{(2(-a^2+b^2))\,\text{Subst}\left(\int \frac{1}{a+b+(a-b)x^2}\,dx, x, \tan\left(\frac{x}{2}\right)\right)}{a^2}$$

$$= -\frac{2\sqrt{a-b}\sqrt{a+b}\arctan\left(\frac{\sqrt{a-b}\tan\left(\frac{x}{2}\right)}{\sqrt{a+b}}\right)}{a^2} - \frac{b\text{arctanh}(\sin(x))}{a^2} + \frac{\tan(x)}{a}$$

Mathematica [A] (verified)

Time = 0.14 (sec) , antiderivative size = 85, normalized size of antiderivative = 1.39

$$\int \frac{\tan^2(x)}{a+b\cos(x)}\,dx$$
$$= \frac{-2\sqrt{-a^2+b^2}\text{arctanh}\left(\frac{(a-b)\tan\left(\frac{x}{2}\right)}{\sqrt{-a^2+b^2}}\right) + b\left(\log\left(\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right) - \log\left(\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right)\right) + a\tan(x)}{a^2}$$

```
[In] Integrate[Tan[x]^2/(a + b*Cos[x]),x]
```

```
[Out] (-2*Sqrt[-a^2 + b^2]*ArcTanh[((a - b)*Tan[x/2])/Sqrt[-a^2 + b^2]] + b*(Log[
Cos[x/2] - Sin[x/2]] - Log[Cos[x/2] + Sin[x/2]]) + a*Tan[x])/a^2
```

Maple [A] (verified)

Time = 0.64 (sec) , antiderivative size = 100, normalized size of antiderivative = 1.64

method	result	size
default	$-\frac{1}{a\left(\tan\left(\frac{x}{2}\right)-1\right)} + \frac{b\ln\left(\tan\left(\frac{x}{2}\right)-1\right)}{a^2} - \frac{1}{a\left(\tan\left(\frac{x}{2}\right)+1\right)} - \frac{b\ln\left(\tan\left(\frac{x}{2}\right)+1\right)}{a^2} + \frac{2\left(-a^2+b^2\right)\arctan\left(\frac{(a-b)\tan\left(\frac{x}{2}\right)}{\sqrt{(a-b)(a+b)}}\right)}{a^2\sqrt{(a-b)(a+b)}}$	100
risch	$\frac{2i}{a\left(\mathrm{e}^{2ix}+1\right)} - \frac{b\ln\left(\mathrm{e}^{ix}+i\right)}{a^2} + \frac{b\ln\left(\mathrm{e}^{ix}-i\right)}{a^2} - \frac{\sqrt{-a^2+b^2}\,\ln\left(\mathrm{e}^{ix}-\frac{i\sqrt{-a^2+b^2}-a}{b}\right)}{a^2} + \frac{\sqrt{-a^2+b^2}\,\ln\left(\mathrm{e}^{ix}+\frac{i\sqrt{-a^2+b^2}+a}{b}\right)}{a^2}$	134

```
[In] int(tan(x)^2/(a+cos(x)*b),x,method=_RETURNVERBOSE)
```

```
[Out] -1/a/(tan(1/2*x)-1)+b/a^2*ln(tan(1/2*x)-1)-1/a/(tan(1/2*x)+1)-b/a^2*ln(tan(
1/2*x)+1)+2/a^2*(-a^2+b^2)/((a-b)*(a+b))^(1/2)*arctan((a-b)*tan(1/2*x)/((a-
b)*(a+b))^(1/2))
```

Fricas [A] (verification not implemented)

none

Time = 0.30 (sec) , antiderivative size = 203, normalized size of antiderivative = 3.33

$$\int \frac{\tan^2(x)}{a+b\cos(x)}\,dx = \left[-\frac{b\cos(x)\log(\sin(x)+1) - b\cos(x)\log(-\sin(x)+1) - \sqrt{-a^2+b^2}\cos(x)\log\left(\frac{2\,ab\cos(x)+(2\,a^2-b^2)\cos(x)}{b^2\cos}\right.}{2\,a^2\cos(x)}\right.$$

$$-\frac{b\cos(x)\log(\sin(x)+1) - b\cos(x)\log(-\sin(x)+1) + 2\sqrt{a^2-b^2}\arctan\left(-\frac{a\cos(x)+b}{\sqrt{a^2-b^2}\sin(x)}\right)\cos(x) - 2}{2\,a^2\cos(x)}$$

```
[In] integrate(tan(x)^2/(a+b*cos(x)),x, algorithm="fricas")
```

```
[Out] [-1/2*(b*cos(x)*log(sin(x) + 1) - b*cos(x)*log(-sin(x) + 1) - sqrt(-a^2 + b
^2)*cos(x)*log((2*a*b*cos(x) + (2*a^2 - b^2)*cos(x)^2 + 2*sqrt(-a^2 + b^2)*
(a*cos(x) + b)*sin(x) - a^2 + 2*b^2)/(b^2*cos(x)^2 + 2*a*b*cos(x) + a^2)) -
 2*a*sin(x))/(a^2*cos(x)), -1/2*(b*cos(x)*log(sin(x) + 1) - b*cos(x)*log(-s
in(x) + 1) + 2*sqrt(a^2 - b^2)*arctan(-(a*cos(x) + b)/(sqrt(a^2 - b^2)*sin(
x)))*cos(x) - 2*a*sin(x))/(a^2*cos(x))]
```

Sympy [F]

$$\int \frac{\tan^2(x)}{a+b\cos(x)}\,dx = \int \frac{\tan^2(x)}{a+b\cos(x)}\,dx$$

```
[In] integrate(tan(x)**2/(a+b*cos(x)),x)
```

```
[Out] Integral(tan(x)**2/(a + b*cos(x)), x)
```

Maxima [F(-2)]

Exception generated.

$$\int \frac{\tan^2(x)}{a+b\cos(x)}\,dx = \text{Exception raised: ValueError}$$

```
[In] integrate(tan(x)^2/(a+b*cos(x)),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*b^2-4*a^2>0)', see `assume?` for
more de
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 111 vs. $2(51) = 102$.

Time = 0.33 (sec) , antiderivative size = 111, normalized size of antiderivative = 1.82

$$\int \frac{\tan^2(x)}{a+b\cos(x)}\,dx = -\frac{b\log\left(\left|\tan\left(\frac{1}{2}x\right)+1\right|\right)}{a^2} + \frac{b\log\left(\left|\tan\left(\frac{1}{2}x\right)-1\right|\right)}{a^2} + \frac{2\left(\pi\left\lfloor\frac{x}{2\pi}+\frac{1}{2}\right\rfloor\mathrm{sgn}(-2a+2b)+\arctan\left(-\frac{a\tan\left(\frac{1}{2}x\right)-b\tan\left(\frac{1}{2}x\right)}{\sqrt{a^2-b^2}}\right)\right)\sqrt{a^2-b^2}}{a^2} - \frac{2\tan\left(\frac{1}{2}x\right)}{\left(\tan\left(\frac{1}{2}x\right)^2-1\right)a}$$

```
[In] integrate(tan(x)^2/(a+b*cos(x)),x, algorithm="giac")
```

```
[Out] -b*log(abs(tan(1/2*x) + 1))/a^2 + b*log(abs(tan(1/2*x) - 1))/a^2 + 2*(pi*fl
oor(1/2*x/pi + 1/2)*sgn(-2*a + 2*b) + arctan(-(a*tan(1/2*x) - b*tan(1/2*x))
/sqrt(a^2 - b^2)))*sqrt(a^2 - b^2)/a^2 - 2*tan(1/2*x)/((tan(1/2*x)^2 - 1)*a
)
```

Mupad [B] (verification not implemented)

Time = 14.26 (sec) , antiderivative size = 77, normalized size of antiderivative = 1.26

$$\int \frac{\tan^2(x)}{a+b\cos(x)}\,dx = \frac{2\,\mathrm{atanh}\left(\frac{\sin\left(\frac{x}{2}\right)\sqrt{b^2-a^2}}{a\,\cos\left(\frac{x}{2}\right)+b\,\cos\left(\frac{x}{2}\right)}\right)\sqrt{b^2-a^2}}{a^2} - \frac{2\,b\,\mathrm{atanh}\left(\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}\right)}{a^2} + \frac{\sin(x)}{a\,\cos(x)}$$

```
[In] int(tan(x)^2/(a + b*cos(x)),x)
```

```
[Out] (2*atanh((sin(x/2)*(b^2 - a^2)^(1/2))/(a*cos(x/2) + b*cos(x/2)))*(b^2 - a^2
)^(1/2))/a^2 - (2*b*atanh(sin(x/2)/cos(x/2)))/a^2 + sin(x)/(a*cos(x))
```

3.13 $\int \frac{\tan(x)}{a+b\cos(x)} \, dx$

Optimal result

Integrand size = 11, antiderivative size = 20

$$\int \frac{\tan(x)}{a+b\cos(x)} \, dx = -\frac{\log(\cos(x))}{a} + \frac{\log(a+b\cos(x))}{a}$$

```
[Out] -ln(cos(x))/a+ln(a+b*cos(x))/a
```

Rubi [A] (verified)

Time = 0.07 (sec) , antiderivative size = 20, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.364, Rules used = {2800, 36, 29, 31}

$$\int \frac{\tan(x)}{a+b\cos(x)} \, dx = \frac{\log(a+b\cos(x))}{a} - \frac{\log(\cos(x))}{a}$$

```
[In] Int[Tan[x]/(a + b*Cos[x]),x]
[Out] -(Log[Cos[x]]/a) + Log[a + b*Cos[x]]/a
```

Rule 29

```
Int[(x_)^(-1), x_Symbol] :> Simp[Log[x], x]
```

Rule 31

```
Int[((a_) + (b_.)*(x_))^(-1), x_Symbol] :> Simp[Log[RemoveContent[a + b*x,
x]]/b, x] /; FreeQ[{a, b}, x]
```

Rule 36

```
Int[1/(((a_.) + (b_.)*(x_))*((c_.) + (d_.)*(x_))), x_Symbol] :> Dist[b/(b*c
 - a*d), Int[1/(a + b*x), x], x] - Dist[d/(b*c - a*d), Int[1/(c + d*x), x],
 x] /; FreeQ[{a, b, c, d}, x] && NeQ[b*c - a*d, 0]
```

Rule 2800

```
Int[((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m_.)*tan[(e_.) + (f_.)*(x_)]^(p
_.), x_Symbol] :> Dist[1/f, Subst[Int[(x^p*(a + x)^m)/(b^2 - x^2)^((p + 1)/
2), x], x, b*Sin[e + f*x]], x] /; FreeQ[{a, b, e, f, m}, x] && NeQ[a^2 - b^
2, 0] && IntegerQ[(p + 1)/2]
```

Rubi steps

$$\begin{aligned} \text{integral} &= -\text{Subst}\left(\int \frac{1}{x(a+x)}\,dx, x, b\cos(x)\right) \\ &= -\frac{\text{Subst}\left(\int \frac{1}{x}\,dx, x, b\cos(x)\right)}{a} + \frac{\text{Subst}\left(\int \frac{1}{a+x}\,dx, x, b\cos(x)\right)}{a} \\ &= -\frac{\log(\cos(x))}{a} + \frac{\log(a+b\cos(x))}{a} \end{aligned}$$

Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 20, normalized size of antiderivative = 1.00

$$\int \frac{\tan(x)}{a+b\cos(x)}\,dx = -\frac{\log(\cos(x))}{a} + \frac{\log(a+b\cos(x))}{a}$$

```
[In] Integrate[Tan[x]/(a + b*Cos[x]),x]
[Out] -(Log[Cos[x]]/a) + Log[a + b*Cos[x]]/a
```

Maple [A] (verified)

Time = 0.50 (sec) , antiderivative size = 21, normalized size of antiderivative = 1.05

method	result	size
default	$-\frac{\ln(\cos(x))}{a}+\frac{\ln(a+\cos(x)b)}{a}$	21
risch	$-\frac{\ln\left(e^{2ix}+1\right)}{a}+\frac{\ln\left(e^{2ix}+\frac{2a\,e^{ix}}{b}+1\right)}{a}$	38

```
[In] int(tan(x)/(a+cos(x)*b),x,method=_RETURNVERBOSE)
[Out] -ln(cos(x))/a+ln(a+cos(x)*b)/a
```

Fricas [A] (verification not implemented)

none

Time = 0.26 (sec) , antiderivative size = 22, normalized size of antiderivative = 1.10

$$\int \frac{\tan(x)}{a+b\cos(x)}\,dx = \frac{\log\left(-b\cos\left(x\right)-a\right)-\log\left(-\cos\left(x\right)\right)}{a}$$

```
[In] integrate(tan(x)/(a+b*cos(x)),x, algorithm="fricas")
[Out] (log(-b*cos(x) - a) - log(-cos(x)))/a
```

Sympy [F]

$$\int \frac{\tan(x)}{a+b\cos(x)}\,dx = \int \frac{\tan\left(x\right)}{a+b\cos\left(x\right)}\,dx$$

```
[In] integrate(tan(x)/(a+b*cos(x)),x)
[Out] Integral(tan(x)/(a + b*cos(x)), x)
```

Maxima [A] (verification not implemented)

none

Time = 0.22 (sec) , antiderivative size = 20, normalized size of antiderivative = 1.00

$$\int \frac{\tan(x)}{a+b\cos(x)}\,dx = \frac{\log\left(b\cos\left(x\right)+a\right)}{a} - \frac{\log\left(\cos\left(x\right)\right)}{a}$$

```
[In] integrate(tan(x)/(a+b*cos(x)),x, algorithm="maxima")
[Out] log(b*cos(x) + a)/a - log(cos(x))/a
```

Giac [A] (verification not implemented)

none

Time = 0.31 (sec) , antiderivative size = 22, normalized size of antiderivative = 1.10

$$\int \frac{\tan(x)}{a+b\cos(x)}\,dx = \frac{\log\left(|b\cos\left(x\right)+a|\right)}{a} - \frac{\log\left(|\cos\left(x\right)|\right)}{a}$$

```
[In] integrate(tan(x)/(a+b*cos(x)),x, algorithm="giac")
[Out] log(abs(b*cos(x) + a))/a - log(abs(cos(x)))/a
```

Mupad [B] (verification not implemented)

Time = 13.92 (sec) , antiderivative size = 48, normalized size of antiderivative = 2.40

$$\int \frac{\tan(x)}{a+b\cos(x)}\,dx = \frac{\operatorname{atan}\left(\frac{a\sin\left(\frac{x}{2}\right)^2}{a\cos\left(\frac{x}{2}\right)^2\,1\mathrm{i}+b\cos\left(\frac{x}{2}\right)^2\,1\mathrm{i}-b\sin\left(\frac{x}{2}\right)^2\,1\mathrm{i}}\right)\,2\mathrm{i}}{a}$$

```
[In] int(tan(x)/(a + b*cos(x)),x)
```

```
[Out] (atan((a*sin(x/2)^2)/(a*cos(x/2)^2*1i + b*cos(x/2)^2*1i - b*sin(x/2)^2*1i))
*2i)/a
```

3.14 $\int \frac{\cot(x)}{a+b\cos(x)}\,dx$

Optimal result

Integrand size = 11, antiderivative size = 54

$$\int \frac{\cot(x)}{a+b\cos(x)}\,dx = \frac{\log(1-\cos(x))}{2(a+b)} + \frac{\log(1+\cos(x))}{2(a-b)} - \frac{a\log(a+b\cos(x))}{a^2-b^2}$$

```
[Out] 1/2*ln(1-cos(x))/(a+b)+1/2*ln(1+cos(x))/(a-b)-a*ln(a+b*cos(x))/(a^2-b^2)
```

Rubi [A] (verified)

Time = 0.11 (sec) , antiderivative size = 54, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.182, Rules used = {2800, 815}

$$\int \frac{\cot(x)}{a+b\cos(x)}\,dx = -\frac{a\log(a+b\cos(x))}{a^2-b^2} + \frac{\log(1-\cos(x))}{2(a+b)} + \frac{\log(\cos(x)+1)}{2(a-b)}$$

```
[In] Int[Cot[x]/(a + b*Cos[x]),x]
```

```
[Out] Log[1 - Cos[x]]/(2*(a + b)) + Log[1 + Cos[x]]/(2*(a - b)) - (a*Log[a + b*Co
s[x]])/(a^2 - b^2)
```

Rule 815

```
Int[(((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_)))/((a_) + (c_.)*(x_)^2),
 x_Symbol] :> Int[ExpandIntegrand[(d + e*x)^m*((f + g*x)/(a + c*x^2)), x],
x] /; FreeQ[{a, c, d, e, f, g}, x] && NeQ[c*d^2 + a*e^2, 0] && IntegerQ[m]
```

Rule 2800

```
Int[((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m_.)*tan[(e_.) + (f_.)*(x_)]^(p
_.), x_Symbol] :> Dist[1/f, Subst[Int[(x^p*(a + x)^m)/(b^2 - x^2)^((p + 1)/
```

```
2), x], x, b*Sin[e + f*x]], x] /; FreeQ[{a, b, e, f, m}, x] && NeQ[a^2 - b^
2, 0] && IntegerQ[(p + 1)/2]
```

Rubi steps

$$\begin{aligned}\text{integral} &= -\text{Subst}\left(\int \frac{x}{(a+x)\,(b^2-x^2)}\,dx, x, b\cos(x)\right)\\ &= -\text{Subst}\left(\int \left(\frac{1}{2(a+b)(b-x)}+\frac{a}{(a-b)(a+b)(a+x)}-\frac{1}{2(a-b)(b+x)}\right)dx, x, b\cos(x)\right)\\ &= \frac{\log(1-\cos(x))}{2(a+b)}+\frac{\log(1+\cos(x))}{2(a-b)}-\frac{a\log(a+b\cos(x))}{a^2-b^2}\end{aligned}$$

Mathematica [A] (verified)

Time = 0.06 (sec) , antiderivative size = 50, normalized size of antiderivative = 0.93

$$\int \frac{\cot(x)}{a+b\cos(x)}\,dx = \frac{\log\left(\cos\left(\frac{x}{2}\right)\right)}{a-b}-\frac{a\log(a+b\cos(x))}{a^2-b^2}+\frac{\log\left(\sin\left(\frac{x}{2}\right)\right)}{a+b}$$

```
[In] Integrate[Cot[x]/(a + b*Cos[x]),x]
```

```
[Out] Log[Cos[x/2]]/(a - b) - (a*Log[a + b*Cos[x]])/(a^2 - b^2) + Log[Sin[x/2]]/(
a + b)
```

Maple [A] (verified)

Time = 0.67 (sec) , antiderivative size = 54, normalized size of antiderivative = 1.00

method	result	size
default	$\frac{\ln(\cos(x)-1)}{2a+2b}-\frac{a\ln(a+\cos(x)b)}{(a+b)(a-b)}+\frac{\ln(\cos(x)+1)}{2a-2b}$	54
risch	$-\frac{ix}{a-b}-\frac{ix}{a+b}+\frac{2ixa}{a^2-b^2}+\frac{\ln(\mathrm{e}^{ix}+1)}{a-b}+\frac{\ln(\mathrm{e}^{ix}-1)}{a+b}-\frac{a\ln\left(\mathrm{e}^{2ix}+\frac{2a\,\mathrm{e}^{ix}}{b}+1\right)}{a^2-b^2}$	101

```
[In] int(cot(x)/(a+cos(x)*b),x,method=_RETURNVERBOSE)
```

```
[Out] 1/(2*a+2*b)*ln(cos(x)-1)-a/(a+b)/(a-b)*ln(a+cos(x)*b)+1/(2*a-2*b)*ln(cos(x)
+1)
```

Fricas [A] (verification not implemented)

none

Time = 0.26 (sec) , antiderivative size = 53, normalized size of antiderivative = 0.98

$$\int \frac{\cot(x)}{a+b\cos(x)}\,dx = -\frac{2\,a\log\left(-b\cos\left(x\right)-a\right)-\left(a+b\right)\log\left(\frac{1}{2}\,\cos\left(x\right)+\frac{1}{2}\right)-\left(a-b\right)\log\left(-\frac{1}{2}\,\cos\left(x\right)+\frac{1}{2}\right)}{2\left(a^2-b^2\right)}$$

```
[In] integrate(cot(x)/(a+b*cos(x)),x, algorithm="fricas")
[Out] -1/2*(2*a*log(-b*cos(x) - a) - (a + b)*log(1/2*cos(x) + 1/2) - (a - b)*log(
-1/2*cos(x) + 1/2))/(a^2 - b^2)
```

Sympy [F]

$$\int \frac{\cot(x)}{a+b\cos(x)}\,dx = \int \frac{\cot\left(x\right)}{a+b\cos\left(x\right)}\,dx$$

```
[In] integrate(cot(x)/(a+b*cos(x)),x)
[Out] Integral(cot(x)/(a + b*cos(x)), x)
```

Maxima [A] (verification not implemented)

none

Time = 0.22 (sec) , antiderivative size = 48, normalized size of antiderivative = 0.89

$$\int \frac{\cot(x)}{a+b\cos(x)}\,dx = -\frac{a\log\left(b\cos\left(x\right)+a\right)}{a^2-b^2}+\frac{\log\left(\cos\left(x\right)+1\right)}{2\left(a-b\right)}+\frac{\log\left(\cos\left(x\right)-1\right)}{2\left(a+b\right)}$$

```
[In] integrate(cot(x)/(a+b*cos(x)),x, algorithm="maxima")
[Out] -a*log(b*cos(x) + a)/(a^2 - b^2) + 1/2*log(cos(x) + 1)/(a - b) + 1/2*log(co
s(x) - 1)/(a + b)
```

Giac [A] (verification not implemented)

none

Time = 0.32 (sec) , antiderivative size = 54, normalized size of antiderivative = 1.00

$$\int \frac{\cot(x)}{a+b\cos(x)}\,dx = -\frac{ab\log(|b\cos(x)+a|)}{a^2b-b^3} + \frac{\log(\cos(x)+1)}{2(a-b)} + \frac{\log(-\cos(x)+1)}{2(a+b)}$$

```
[In] integrate(cot(x)/(a+b*cos(x)),x, algorithm="giac")
```

```
[Out] -a*b*log(abs(b*cos(x) + a))/(a^2*b - b^3) + 1/2*log(cos(x) + 1)/(a - b) + 1
/2*log(-cos(x) + 1)/(a + b)
```

Mupad [B] (verification not implemented)

Time = 14.06 (sec) , antiderivative size = 47, normalized size of antiderivative = 0.87

$$\int \frac{\cot(x)}{a+b\cos(x)}\,dx = \frac{\ln\left(\tan\left(\frac{x}{2}\right)\right)}{a+b} - \frac{a\,\ln\left(a+b+a\tan\left(\frac{x}{2}\right)^2-b\tan\left(\frac{x}{2}\right)^2\right)}{a^2-b^2}$$

```
[In] int(cot(x)/(a + b*cos(x)),x)
```

```
[Out] log(tan(x/2))/(a + b) - (a*log(a + b + a*tan(x/2)^2 - b*tan(x/2)^2))/(a^2 -
 b^2)
```

3.15 $\int \frac{\cot^2(x)}{a+b\cos(x)}\,dx$

Optimal result

Integrand size = 13, antiderivative size = 77

$$\int \frac{\cot^2(x)}{a+b\cos(x)}\,dx=-\frac{2a^2\arctan\left(\frac{\sqrt{a-b}\tan\left(\frac{x}{2}\right)}{\sqrt{a+b}}\right)}{(a-b)^{3/2}(a+b)^{3/2}}-\frac{a\cot(x)}{a^2-b^2}+\frac{b\csc(x)}{a^2-b^2}$$

```
[Out] -2*a^2*arctan((a-b)^(1/2)*tan(1/2*x)/(a+b)^(1/2))/(a-b)^(3/2)/(a+b)^(3/2)-a
*cot(x)/(a^2-b^2)+b*csc(x)/(a^2-b^2)
```

Rubi [A] (verified)

Time = 0.19 (sec) , antiderivative size = 77, normalized size of antiderivative = 1.00, number of steps used = 7, number of rules used = 6, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.462, Rules used = {2806, 3852, 8, 2686, 2738, 211}

$$\int \frac{\cot^2(x)}{a+b\cos(x)}\,dx=-\frac{2a^2\arctan\left(\frac{\sqrt{a-b}\tan\left(\frac{x}{2}\right)}{\sqrt{a+b}}\right)}{(a-b)^{3/2}(a+b)^{3/2}}-\frac{a\cot(x)}{a^2-b^2}+\frac{b\csc(x)}{a^2-b^2}$$

```
[In] Int[Cot[x]^2/(a + b*Cos[x]),x]
```

```
[Out] (-2*a^2*ArcTan[(Sqrt[a - b]*Tan[x/2])/Sqrt[a + b]])/((a - b)^(3/2)*(a + b)^
(3/2)) - (a*Cot[x])/(a^2 - b^2) + (b*Csc[x])/(a^2 - b^2)
```

Rule 8

```
Int[a_, x_Symbol] :> Simp[a*x, x] /; FreeQ[a, x]
```

Rule 211

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

Rule 2686

```
Int[((a_.)*sec[(e_.) + (f_.)*(x_)])^(m_.)*((b_.)*tan[(e_.) + (f_.)*(x_)])^(
n_.), x_Symbol] :> Dist[a/f, Subst[Int[(a*x)^(m - 1)*(-1 + x^2)^((n - 1)/2)
, x], x, Sec[e + f*x]], x] /; FreeQ[{a, e, f, m}, x] && IntegerQ[(n - 1)/2]
 &&  !(IntegerQ[m/2] && LtQ[0, m, n + 1])
```

Rule 2738

```
Int[((a_) + (b_.)*sin[Pi/2 + (c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{
e = FreeFactors[Tan[(c + d*x)/2], x]}, Dist[2*(e/d), Subst[Int[1/(a + b + (
a - b)*e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x]
&& NeQ[a^2 - b^2, 0]
```

Rule 2806

```
Int[((g_.)*tan[(e_.) + (f_.)*(x_)])^(p_)/((a_) + (b_.)*sin[(e_.) + (f_.)*(x
_)]), x_Symbol] :> Dist[a/(a^2 - b^2), Int[(g*Tan[e + f*x])^p/Sin[e + f*x]^
2, x], x] + (-Dist[b*(g/(a^2 - b^2)), Int[(g*Tan[e + f*x])^(p - 1)/Cos[e +
f*x], x], x] - Dist[a^2*(g^2/(a^2 - b^2)), Int[(g*Tan[e + f*x])^(p - 2)/(a
+ b*Sin[e + f*x]), x], x]) /; FreeQ[{a, b, e, f, g}, x] && NeQ[a^2 - b^2, 0
] && IntegersQ[2*p] && GtQ[p, 1]
```

Rule 3852

```
Int[csc[(c_.) + (d_.)*(x_)]^(n_), x_Symbol] :> Dist[-d^(-1), Subst[Int[Expa
ndIntegrand[(1 + x^2)^(n/2 - 1), x], x], x, Cot[c + d*x]], x] /; FreeQ[{c,
d}, x] && IGtQ[n/2, 0]
```

Rubi steps

$$\begin{aligned}
\text{integral} &= \frac{a\int \csc^2(x)\,dx}{a^2-b^2}-\frac{a^2\int \frac{1}{a+b\cos(x)}\,dx}{a^2-b^2}-\frac{b\int \cot(x)\csc(x)\,dx}{a^2-b^2}\\
&= -\frac{a\text{Subst}\left(\int 1\,dx,x,\cot(x)\right)}{a^2-b^2}-\frac{\left(2a^2\right)\text{Subst}\left(\int \frac{1}{a+b+(a-b)x^2}\,dx,x,\tan\left(\frac{x}{2}\right)\right)}{a^2-b^2}\\
&\quad+\frac{b\text{Subst}\left(\int 1\,dx,x,\csc(x)\right)}{a^2-b^2}\\
&= -\frac{2a^2\arctan\left(\frac{\sqrt{a-b}\tan\left(\frac{x}{2}\right)}{\sqrt{a+b}}\right)}{(a-b)^{3/2}(a+b)^{3/2}}-\frac{a\cot(x)}{a^2-b^2}+\frac{b\csc(x)}{a^2-b^2}
\end{aligned}$$

Mathematica [A] (verified)

Time = 0.23 (sec) , antiderivative size = 67, normalized size of antiderivative = 0.87

$$\int \frac{\cot^2(x)}{a+b\cos(x)}\,dx = -\frac{2a^2\text{arctanh}\left(\frac{(a-b)\tan\left(\frac{x}{2}\right)}{\sqrt{-a^2+b^2}}\right)}{(-a^2+b^2)^{3/2}} + \frac{-a\cot(x)+b\csc(x)}{a^2-b^2}$$

```
[In] Integrate[Cot[x]^2/(a + b*Cos[x]),x]
```

```
[Out] (-2*a^2*ArcTanh[((a - b)*Tan[x/2])/Sqrt[-a^2 + b^2]])/(-a^2 + b^2)^(3/2) +
(-(a*Cot[x]) + b*Csc[x])/(a^2 - b^2)
```

Maple [A] (verified)

Time = 0.83 (sec) , antiderivative size = 78, normalized size of antiderivative = 1.01

method	result	size
default	$\frac{\tan\left(\frac{x}{2}\right)}{2a-2b}-\frac{1}{2(a+b)\tan\left(\frac{x}{2}\right)}-\frac{2a^2\arctan\left(\frac{(a-b)\tan\left(\frac{x}{2}\right)}{\sqrt{(a-b)(a+b)}}\right)}{(a-b)(a+b)\sqrt{(a-b)(a+b)}}$	78
risch	$-\frac{2i(-e^{ix}b+a)}{(e^{2ix}-1)(a^2-b^2)}+\frac{a^2\ln\left(e^{ix}+\frac{ia^2-ib^2+a\sqrt{-a^2+b^2}}{b\sqrt{-a^2+b^2}}\right)}{\sqrt{-a^2+b^2}\,(a+b)(a-b)}-\frac{a^2\ln\left(e^{ix}+\frac{-ia^2+ib^2+a\sqrt{-a^2+b^2}}{\sqrt{-a^2+b^2}\,b}\right)}{\sqrt{-a^2+b^2}\,(a+b)(a-b)}$	186

```
[In] int(cot(x)^2/(a+cos(x)*b),x,method=_RETURNVERBOSE)
```

```
[Out] 1/2*tan(1/2*x)/(a-b)-1/2/(a+b)/tan(1/2*x)-2/(a-b)/(a+b)*a^2/((a-b)*(a+b))^(
1/2)*arctan((a-b)*tan(1/2*x)/((a-b)*(a+b))^(1/2))
```

Fricas [A] (verification not implemented)

none

Time = 0.28 (sec) , antiderivative size = 230, normalized size of antiderivative = 2.99

$$\int \frac{\cot^2(x)}{a+b\cos(x)}\,dx = \left[\frac{\sqrt{-a^2+b^2}a^2\log\left(\frac{2\,ab\cos(x)+(2\,a^2-b^2)\cos(x)^2+2\,\sqrt{-a^2+b^2}(a\cos(x)+b)\sin(x)-a^2+2\,b^2}{b^2\cos(x)^2+2\,ab\cos(x)+a^2}\right)\sin\left(x\right)+2\,a^2b-2\,b^3-2\,(a^3-}{2\,(a^4-2\,a^2b^2+b^4)\sin\left(x\right)}\right. \\ \left. -\frac{\sqrt{a^2-b^2}a^2\arctan\left(-\frac{a\cos(x)+b}{\sqrt{a^2-b^2}\sin(x)}\right)\sin\left(x\right)-a^2b+b^3+(a^3-ab^2)\cos\left(x\right)}{(a^4-2\,a^2b^2+b^4)\sin\left(x\right)}\right]$$

```
[In] integrate(cot(x)^2/(a+b*cos(x)),x, algorithm="fricas")
```

```
[Out] [1/2*(sqrt(-a^2 + b^2)*a^2*log((2*a*b*cos(x) + (2*a^2 - b^2)*cos(x)^2 + 2*s
qrt(-a^2 + b^2)*(a*cos(x) + b)*sin(x) - a^2 + 2*b^2)/(b^2*cos(x)^2 + 2*a*b*
cos(x) + a^2))*sin(x) + 2*a^2*b - 2*b^3 - 2*(a^3 - a*b^2)*cos(x))/((a^4 - 2
*a^2*b^2 + b^4)*sin(x)), -(sqrt(a^2 - b^2)*a^2*arctan(-(a*cos(x) + b)/(sqrt
(a^2 - b^2)*sin(x)))*sin(x) - a^2*b + b^3 + (a^3 - a*b^2)*cos(x))/((a^4 - 2
*a^2*b^2 + b^4)*sin(x))]
```

Sympy [F]

$$\int \frac{\cot^2(x)}{a+b\cos(x)}\,dx = \int \frac{\cot^2{(x)}}{a+b\cos{(x)}}\,dx$$

```
[In] integrate(cot(x)**2/(a+b*cos(x)),x)
```

```
[Out] Integral(cot(x)**2/(a + b*cos(x)), x)
```

Maxima [F(-2)]

Exception generated.

$$\int \frac{\cot^2(x)}{a+b\cos(x)}\,dx = \text{Exception raised: ValueError}$$

```
[In] integrate(cot(x)^2/(a+b*cos(x)),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*b^2-4*a^2>0)', see `assume?` for
more de
```

Giac [A] (verification not implemented)

none

Time = 0.33 (sec) , antiderivative size = 91, normalized size of antiderivative = 1.18

$$\int \frac{\cot^2(x)}{a+b\cos(x)}\,dx = \frac{2\left(\pi\left\lfloor\frac{x}{2\pi}+\frac{1}{2}\right\rfloor\operatorname{sgn}(-2\,a+2\,b)+\arctan\left(-\frac{a\tan\left(\frac{1}{2}\,x\right)-b\tan\left(\frac{1}{2}\,x\right)}{\sqrt{a^2-b^2}}\right)\right)a^2}{(a^2-b^2)^{\frac{3}{2}}} + \frac{\tan\left(\frac{1}{2}\,x\right)}{2\,(a-b)} - \frac{1}{2\,(a+b)\tan\left(\frac{1}{2}\,x\right)}$$

```
[In] integrate(cot(x)^2/(a+b*cos(x)),x, algorithm="giac")
```

```
[Out] 2*(pi*floor(1/2*x/pi + 1/2)*sgn(-2*a + 2*b) + arctan(-(a*tan(1/2*x) - b*tan
(1/2*x))/sqrt(a^2 - b^2)))*a^2/(a^2 - b^2)^(3/2) + 1/2*tan(1/2*x)/(a - b) -
 1/2/((a + b)*tan(1/2*x))
```

Mupad [B] (verification not implemented)

Time = 14.71 (sec) , antiderivative size = 86, normalized size of antiderivative = 1.12

$$\int \frac{\cot^2(x)}{a+b\cos(x)}\,dx=\frac{\tan\left(\frac{x}{2}\right)}{2\,a-2\,b}-\frac{2\,a^2\,\mathrm{atan}\left(\frac{\tan\left(\frac{x}{2}\right)\,\left(a^2-b^2\right)}{(a+b)^{3/2}\,\sqrt{a-b}}\right)}{(a+b)^{3/2}\,(a-b)^{3/2}}-\frac{a-b}{\tan\left(\frac{x}{2}\right)\,(a+b)\,(2\,a-2\,b)}$$

```
[In] int(cot(x)^2/(a + b*cos(x)),x)
```

```
[Out] tan(x/2)/(2*a - 2*b) - (2*a^2*atan((tan(x/2)*(a^2 - b^2))/((a + b)^(3/2)*(a
 - b)^(1/2))))/((a + b)^(3/2)*(a - b)^(3/2)) - (a - b)/(tan(x/2)*(a + b)*(2
*a - 2*b))
```

3.16 $\int \frac{\cot^3(x)}{a+b\cos(x)}\,dx$

Optimal result

Integrand size = 13, antiderivative size = 93

$$\int \frac{\cot^3(x)}{a+b\cos(x)}\,dx = -\frac{(a-b\cos(x))\csc^2(x)}{2(a^2-b^2)} - \frac{(2a+b)\log(1-\cos(x))}{4(a+b)^2} - \frac{(2a-b)\log(1+\cos(x))}{4(a-b)^2} + \frac{a^3\log(a+b\cos(x))}{(a^2-b^2)^2}$$

```
[Out] -1/2*(a-b*cos(x))*csc(x)^2/(a^2-b^2)-1/4*(2*a+b)*ln(1-cos(x))/(a+b)^2-1/4*(
2*a-b)*ln(1+cos(x))/(a-b)^2+a^3*ln(a+b*cos(x))/(a^2-b^2)^2
```

Rubi [A] (verified)

Time = 0.21 (sec) , antiderivative size = 93, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.231, Rules used = {2800, 1661, 815}

$$\int \frac{\cot^3(x)}{a+b\cos(x)}\,dx = -\frac{\csc^2(x)(a-b\cos(x))}{2(a^2-b^2)} + \frac{a^3\log(a+b\cos(x))}{(a^2-b^2)^2} - \frac{(2a+b)\log(1-\cos(x))}{4(a+b)^2} - \frac{(2a-b)\log(\cos(x)+1)}{4(a-b)^2}$$

```
[In] Int[Cot[x]^3/(a + b*Cos[x]),x]
```

```
[Out] -1/2*((a - b*Cos[x])*Csc[x]^2)/(a^2 - b^2) - ((2*a + b)*Log[1 - Cos[x]])/(4
*(a + b)^2) - ((2*a - b)*Log[1 + Cos[x]])/(4*(a - b)^2) + (a^3*Log[a + b*Co
s[x]])/(a^2 - b^2)^2
```

Rule 815

```
Int[(((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_)))/((a_) + (c_.)*(x_)^2),
 x_Symbol] :> Int[ExpandIntegrand[(d + e*x)^m*((f + g*x)/(a + c*x^2)), x],
x] /; FreeQ[{a, c, d, e, f, g}, x] && NeQ[c*d^2 + a*e^2, 0] && IntegerQ[m]
```

Rule 1661

```
Int[(Pq_)*((d_) + (e_.)*(x_))^(m_.)*((a_) + (c_.)*(x_)^2)^(p_), x_Symbol] :
> With[{Q = PolynomialQuotient[(d + e*x)^m*Pq, a + c*x^2, x], f = Coeff[Pol
ynomialRemainder[(d + e*x)^m*Pq, a + c*x^2, x], x, 0], g = Coeff[Polynomial
Remainder[(d + e*x)^m*Pq, a + c*x^2, x], x, 1]}, Simp[(a*g - c*f*x)*((a + c
*x^2)^(p + 1)/(2*a*c*(p + 1))), x] + Dist[1/(2*a*c*(p + 1)), Int[(d + e*x)^
m*(a + c*x^2)^(p + 1)*ExpandToSum[(2*a*c*(p + 1)*Q)/(d + e*x)^m + (c*f*(2*p
 + 3))/(d + e*x)^m, x], x], x]] /; FreeQ[{a, c, d, e}, x] && PolyQ[Pq, x] &
& NeQ[c*d^2 + a*e^2, 0] && LtQ[p, -1] && ILtQ[m, 0]
```

Rule 2800

```
Int[((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m_.)*tan[(e_.) + (f_.)*(x_)]^(p
_.), x_Symbol] :> Dist[1/f, Subst[Int[(x^p*(a + x)^m)/(b^2 - x^2)^((p + 1)/
2), x], x, b*Sin[e + f*x]], x] /; FreeQ[{a, b, e, f, m}, x] && NeQ[a^2 - b^
2, 0] && IntegerQ[(p + 1)/2]
```

Rubi steps

$$\begin{aligned}
\text{integral} &= -\text{Subst}\left(\int \frac{x^3}{(a+x)\left(b^2-x^2\right)^2}\,dx, x, b\cos(x)\right)\\
&= -\frac{(a-b\cos(x))\csc^2(x)}{2\left(a^2-b^2\right)} - \frac{\text{Subst}\left(\int \frac{\frac{ab^4}{a^2-b^2}-\frac{b^2\left(2a^2-b^2\right)x}{a^2-b^2}}{(a+x)\left(b^2-x^2\right)}\,dx, x, b\cos(x)\right)}{2b^2}\\
&= -\frac{(a-b\cos(x))\csc^2(x)}{2\left(a^2-b^2\right)}\\
&\quad - \frac{\text{Subst}\left(\int \left(-\frac{b^2(2a+b)}{2(a+b)^2(b-x)} - \frac{2a^3b^2}{(a-b)^2(a+b)^2(a+x)} + \frac{(2a-b)b^2}{2(a-b)^2(b+x)}\right)dx, x, b\cos(x)\right)}{2b^2}\\
&= -\frac{(a-b\cos(x))\csc^2(x)}{2\left(a^2-b^2\right)} - \frac{(2a+b)\log(1-\cos(x))}{4(a+b)^2}\\
&\quad - \frac{(2a-b)\log(1+\cos(x))}{4(a-b)^2} + \frac{a^3\log(a+b\cos(x))}{\left(a^2-b^2\right)^2}
\end{aligned}$$

Mathematica [A] (verified)

Time = 0.38 (sec) , antiderivative size = 100, normalized size of antiderivative = 1.08

$$\int \frac{\cot^3(x)}{a+b\cos(x)}\,dx = \frac{1}{8}\left(-\frac{\csc^2\left(\frac{x}{2}\right)}{a+b}+\frac{4(-2a+b)\log\left(\cos\left(\frac{x}{2}\right)\right)}{(a-b)^2}+\frac{8a^3\log(a+b\cos(x))}{\left(a^2-b^2\right)^2} -\frac{4(2a+b)\log\left(\sin\left(\frac{x}{2}\right)\right)}{(a+b)^2}-\frac{\sec^2\left(\frac{x}{2}\right)}{a-b}\right)$$

```
[In] Integrate[Cot[x]^3/(a + b*Cos[x]),x]
```

```
[Out] (-(Csc[x/2]^2/(a + b)) + (4*(-2*a + b)*Log[Cos[x/2]])/(a - b)^2 + (8*a^3*Lo
g[a + b*Cos[x]])/(a^2 - b^2)^2 - (4*(2*a + b)*Log[Sin[x/2]])/(a + b)^2 - Se
c[x/2]^2/(a - b))/8
```

Maple [A] (verified)

Time = 0.90 (sec) , antiderivative size = 96, normalized size of antiderivative = 1.03

method	result
default	$\frac{a^3\ln(a+\cos(x)b)}{(a+b)^2(a-b)^2}-\frac{1}{(4a-4b)(\cos(x)+1)}+\frac{(-2a+b)\ln(\cos(x)+1)}{4(a-b)^2}+\frac{1}{(4a+4b)(\cos(x)-1)}+\frac{(-2a-b)\ln(\cos(x)-1)}{4(a+b)^2}$
risch	$\frac{ixa}{a^2-2ab+b^2}-\frac{ixb}{2(a^2-2ab+b^2)}+\frac{ixa}{a^2+2ab+b^2}+\frac{ixb}{2a^2+4ab+2b^2}-\frac{2ix\,a^3}{a^4-2a^2b^2+b^4}-\frac{e^{3ix}b-2a\,e^{2ix}+e^{ix}b}{(e^{2ix}-1)^2(a^2-b^2)}-\frac{\ln(e^{ix}+1)a}{a^2-2ab+b^2}+\frac{\ln}{2a^2}$

```
[In] int(cot(x)^3/(a+cos(x)*b),x,method=_RETURNVERBOSE)
```

```
[Out] a^3/(a+b)^2/(a-b)^2*ln(a+cos(x)*b)-1/(4*a-4*b)/(cos(x)+1)+1/4/(a-b)^2*(-2*a
+b)*ln(cos(x)+1)+1/(4*a+4*b)/(cos(x)-1)+1/4/(a+b)^2*(-2*a-b)*ln(cos(x)-1)
```

Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 185 vs. 2(88) = 176.

Time = 0.32 (sec) , antiderivative size = 185, normalized size of antiderivative = 1.99

$$\int \frac{\cot^3(x)}{a+b\cos(x)}\,dx = -\frac{2\,a^3-2\,ab^2-2\,(a^2b-b^3)\cos(x)+4\,\left(a^3\cos(x)^2-a^3\right)\log(-b\cos(x)-a)+\left(2\,a^3+3\,a^2b-b^3-(2\,a^3}{4\,\left(a^4-2\,a^2b^2+b^4\right.}$$

```
[In] integrate(cot(x)^3/(a+b*cos(x)),x, algorithm="fricas")
```

```
[Out] -1/4*(2*a^3 - 2*a*b^2 - 2*(a^2*b - b^3)*cos(x) + 4*(a^3*cos(x)^2 - a^3)*log
(-b*cos(x) - a) + (2*a^3 + 3*a^2*b - b^3 - (2*a^3 + 3*a^2*b - b^3)*cos(x)^2
```

```
)*log(1/2*cos(x) + 1/2) + (2*a^3 - 3*a^2*b + b^3 - (2*a^3 - 3*a^2*b + b^3)*
cos(x)^2)*log(-1/2*cos(x) + 1/2))/(a^4 - 2*a^2*b^2 + b^4 - (a^4 - 2*a^2*b^2
 + b^4)*cos(x)^2)
```

Sympy [F]

$$\int \frac{\cot^3(x)}{a+b\cos(x)}\,dx = \int \frac{\cot^3(x)}{a+b\cos(x)}\,dx$$

```
[In] integrate(cot(x)**3/(a+b*cos(x)),x)
[Out] Integral(cot(x)**3/(a + b*cos(x)), x)
```

Maxima [A] (verification not implemented)

none

Time = 0.21 (sec) , antiderivative size = 116, normalized size of antiderivative = 1.25

$$\int \frac{\cot^3(x)}{a+b\cos(x)}\,dx = \frac{a^3\log\left(b\cos\left(x\right)+a\right)}{a^4-2\,a^2b^2+b^4} - \frac{(2\,a-b)\log\left(\cos\left(x\right)+1\right)}{4\,(a^2-2\,ab+b^2)} - \frac{(2\,a+b)\log\left(\cos\left(x\right)-1\right)}{4\,(a^2+2\,ab+b^2)} - \frac{b\cos\left(x\right)-a}{2\,\left((a^2-b^2)\cos\left(x\right)^2-a^2+b^2\right)}$$

```
[In] integrate(cot(x)^3/(a+b*cos(x)),x, algorithm="maxima")
[Out] a^3*log(b*cos(x) + a)/(a^4 - 2*a^2*b^2 + b^4) - 1/4*(2*a - b)*log(cos(x) +
1)/(a^2 - 2*a*b + b^2) - 1/4*(2*a + b)*log(cos(x) - 1)/(a^2 + 2*a*b + b^2)
- 1/2*(b*cos(x) - a)/((a^2 - b^2)*cos(x)^2 - a^2 + b^2)
```

Giac [A] (verification not implemented)

none

Time = 0.32 (sec) , antiderivative size = 138, normalized size of antiderivative = 1.48

$$\int \frac{\cot^3(x)}{a+b\cos(x)}\,dx = \frac{a^3b\log\left(|b\cos\left(x\right)+a|\right)}{a^4b-2\,a^2b^3+b^5} - \frac{(2\,a-b)\log\left(\cos\left(x\right)+1\right)}{4\,(a^2-2\,ab+b^2)} - \frac{(2\,a+b)\log\left(-\cos\left(x\right)+1\right)}{4\,(a^2+2\,ab+b^2)} + \frac{a^3-ab^2-(a^2b-b^3)\cos\left(x\right)}{2\,(a+b)^2(a-b)^2(\cos\left(x\right)+1)(\cos\left(x\right)-1)}$$

```
[In] integrate(cot(x)^3/(a+b*cos(x)),x, algorithm="giac")
```

```
[Out] a^3*b*log(abs(b*cos(x) + a))/(a^4*b - 2*a^2*b^3 + b^5) - 1/4*(2*a - b)*log(
cos(x) + 1)/(a^2 - 2*a*b + b^2) - 1/4*(2*a + b)*log(-cos(x) + 1)/(a^2 + 2*a
*b + b^2) + 1/2*(a^3 - a*b^2 - (a^2*b - b^3)*cos(x))/((a + b)^2*(a - b)^2*(
cos(x) + 1)*(cos(x) - 1))
```

Mupad [B] (verification not implemented)

Time = 14.30 (sec) , antiderivative size = 116, normalized size of antiderivative = 1.25

$$\int \frac{\cot^3(x)}{a+b\cos(x)}\,dx = \frac{a^3\,\ln\left(a+b+a\tan\left(\frac{x}{2}\right)^2-b\tan\left(\frac{x}{2}\right)^2\right)}{a^4-2\,a^2\,b^2+b^4}-\frac{\tan\left(\frac{x}{2}\right)^2}{2\,\left(4\,a-4\,b\right)} - \frac{\ln\left(\tan\left(\frac{x}{2}\right)\right)\,\left(2\,a+b\right)}{2\,a^2+4\,a\,b+2\,b^2}-\frac{a-b}{2\tan\left(\frac{x}{2}\right)^2\,\left(a+b\right)\,\left(4\,a-4\,b\right)}$$

```
[In] int(cot(x)^3/(a + b*cos(x)),x)
```

```
[Out] (a^3*log(a + b + a*tan(x/2)^2 - b*tan(x/2)^2))/(a^4 + b^4 - 2*a^2*b^2) - ta
n(x/2)^2/(2*(4*a - 4*b)) - (log(tan(x/2))*(2*a + b))/(4*a*b + 2*a^2 + 2*b^2
) - (a - b)/(2*tan(x/2)^2*(a + b)*(4*a - 4*b))
```

3.17 $\int \frac{\cot^4(x)}{a+b\cos(x)}\,dx$

Optimal result

Integrand size = 13, antiderivative size = 138

$$\int \frac{\cot^4(x)}{a+b\cos(x)}\,dx=\frac{2a^4\arctan\left(\frac{\sqrt{a-b}\tan\left(\frac{x}{2}\right)}{\sqrt{a+b}}\right)}{(a-b)^{5/2}(a+b)^{5/2}}+\frac{a^3\cot(x)}{\left(a^2-b^2\right)^2}-\frac{a\cot^3(x)}{3\left(a^2-b^2\right)}-\frac{a^2b\csc(x)}{\left(a^2-b^2\right)^2}-\frac{b\csc(x)}{a^2-b^2}+\frac{b\csc^3(x)}{3\left(a^2-b^2\right)}$$

```
[Out] 2*a^4*arctan((a-b)^(1/2)*tan(1/2*x)/(a+b)^(1/2))/(a-b)^(5/2)/(a+b)^(5/2)+a^
3*cot(x)/(a^2-b^2)^2-1/3*a*cot(x)^3/(a^2-b^2)-a^2*b*csc(x)/(a^2-b^2)^2-b*cs
c(x)/(a^2-b^2)+1/3*b*csc(x)^3/(a^2-b^2)
```

Rubi [A] (verified)

Time = 0.23 (sec) , antiderivative size = 138, normalized size of antiderivative = 1.00, number of steps used = 12, number of rules used = 8, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.615, Rules used = {2806, 2687, 30, 2686, 3852, 8, 2738, 211}

$$\int \frac{\cot^4(x)}{a+b\cos(x)}\,dx=\frac{2a^4\arctan\left(\frac{\sqrt{a-b}\tan\left(\frac{x}{2}\right)}{\sqrt{a+b}}\right)}{(a-b)^{5/2}(a+b)^{5/2}}-\frac{a\cot^3(x)}{3\left(a^2-b^2\right)}+\frac{b\csc^3(x)}{3\left(a^2-b^2\right)}-\frac{a^2b\csc(x)}{\left(a^2-b^2\right)^2}-\frac{b\csc(x)}{a^2-b^2}+\frac{a^3\cot(x)}{\left(a^2-b^2\right)^2}$$

```
[In] Int[Cot[x]^4/(a + b*Cos[x]),x]
```

```
[Out] (2*a^4*ArcTan[(Sqrt[a - b]*Tan[x/2])/Sqrt[a + b]])/((a - b)^(5/2)*(a + b)^(
5/2)) + (a^3*Cot[x])/(a^2 - b^2)^2 - (a*Cot[x]^3)/(3*(a^2 - b^2)) - (a^2*b*
```

```
Csc[x])/(a^2 - b^2)^2 - (b*Csc[x])/(a^2 - b^2) + (b*Csc[x]^3)/(3*(a^2 - b^2
))
```

Rule 8

```
Int[a_, x_Symbol] :> Simp[a*x, x] /; FreeQ[a, x]
```

Rule 30

```
Int[(x_)^(m_.), x_Symbol] :> Simp[x^(m + 1)/(m + 1), x] /; FreeQ[m, x] && N
eQ[m, -1]
```

Rule 211

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

Rule 2686

```
Int[((a_.)*sec[(e_.) + (f_.)*(x_)])^(m_.)*((b_.)*tan[(e_.) + (f_.)*(x_)])^(
n_.), x_Symbol] :> Dist[a/f, Subst[Int[(a*x)^(m - 1)*(-1 + x^2)^((n - 1)/2)
, x], x, Sec[e + f*x]], x] /; FreeQ[{a, e, f, m}, x] && IntegerQ[(n - 1)/2]
 &&  !(IntegerQ[m/2] && LtQ[0, m, n + 1])
```

Rule 2687

```
Int[sec[(e_.) + (f_.)*(x_)]^(m_)*((b_.)*tan[(e_.) + (f_.)*(x_)])^(n_.), x_S
ymbol] :> Dist[1/f, Subst[Int[(b*x)^n*(1 + x^2)^(m/2 - 1), x], x, Tan[e + f
*x]], x] /; FreeQ[{b, e, f, n}, x] && IntegerQ[m/2] &&  !(IntegerQ[(n - 1)/
2] && LtQ[0, n, m - 1])
```

Rule 2738

```
Int[((a_) + (b_.)*sin[Pi/2 + (c_.) + (d_.)*(x_)])^(-1), x_Symbol] :> With[{
e = FreeFactors[Tan[(c + d*x)/2], x]}, Dist[2*(e/d), Subst[Int[1/(a + b + (
a - b)*e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x]] /; FreeQ[{a, b, c, d}, x]
&& NeQ[a^2 - b^2, 0]
```

Rule 2806

```
Int[((g_.)*tan[(e_.) + (f_.)*(x_)])^(p_)/((a_) + (b_.)*sin[(e_.) + (f_.)*(x
_)]), x_Symbol] :> Dist[a/(a^2 - b^2), Int[(g*Tan[e + f*x])^p/Sin[e + f*x]^
2, x], x] + (-Dist[b*(g/(a^2 - b^2)), Int[(g*Tan[e + f*x])^(p - 1)/Cos[e +
f*x], x], x] - Dist[a^2*(g^2/(a^2 - b^2)), Int[(g*Tan[e + f*x])^(p - 2)/(a
+ b*Sin[e + f*x]), x], x]) /; FreeQ[{a, b, e, f, g}, x] && NeQ[a^2 - b^2, 0
] && IntegersQ[2*p] && GtQ[p, 1]
```

Rule 3852

```
Int[csc[(c_.) + (d_.)*(x_)]^(n_), x_Symbol] :> Dist[-d^(-1), Subst[Int[Expa
ndIntegrand[(1 + x^2)^(n/2 - 1), x], x], x, Cot[c + d*x]], x] /; FreeQ[{c,
d}, x] && IGtQ[n/2, 0]
```

Rubi steps

$$\begin{aligned}
\text{integral} &= \frac{a\int \cot^2(x)\csc^2(x)\,dx}{a^2-b^2}-\frac{a^2\int \frac{\cot^2(x)}{a+b\cos(x)}\,dx}{a^2-b^2}-\frac{b\int \cot^3(x)\csc(x)\,dx}{a^2-b^2}\\
&= -\frac{a^3\int \csc^2(x)\,dx}{(a^2-b^2)^2}+\frac{a^4\int \frac{1}{a+b\cos(x)}\,dx}{(a^2-b^2)^2}+\frac{(a^2b)\int \cot(x)\csc(x)\,dx}{(a^2-b^2)^2}\\
&\quad+\frac{a\text{Subst}\left(\int x^2\,dx,x,-\cot(x)\right)}{a^2-b^2}+\frac{b\text{Subst}\left(\int (-1+x^2)\,dx,x,\csc(x)\right)}{a^2-b^2}\\
&= -\frac{a\cot^3(x)}{3(a^2-b^2)}-\frac{b\csc(x)}{a^2-b^2}+\frac{b\csc^3(x)}{3(a^2-b^2)}+\frac{a^3\text{Subst}(\int 1\,dx,x,\cot(x))}{(a^2-b^2)^2}\\
&\quad+\frac{(2a^4)\,\text{Subst}\left(\int \frac{1}{a+b+(a-b)x^2}\,dx,x,\tan\left(\frac{x}{2}\right)\right)}{(a^2-b^2)^2}-\frac{(a^2b)\,\text{Subst}(\int 1\,dx,x,\csc(x))}{(a^2-b^2)^2}\\
&= \frac{2a^4\arctan\left(\frac{\sqrt{a-b}\tan\left(\frac{x}{2}\right)}{\sqrt{a+b}}\right)}{(a-b)^{5/2}(a+b)^{5/2}}+\frac{a^3\cot(x)}{(a^2-b^2)^2}-\frac{a\cot^3(x)}{3(a^2-b^2)}-\frac{a^2b\csc(x)}{(a^2-b^2)^2}-\frac{b\csc(x)}{a^2-b^2}+\frac{b\csc^3(x)}{3(a^2-b^2)}
\end{aligned}$$

Mathematica [A] (verified)

Time = 0.47 (sec) , antiderivative size = 112, normalized size of antiderivative = 0.81

$$\begin{aligned}
&\int \frac{\cot^4(x)}{a+b\cos(x)}\,dx\\
&= -\frac{2a^4\text{arctanh}\left(\frac{(a-b)\tan\left(\frac{x}{2}\right)}{\sqrt{-a^2+b^2}}\right)}{(-a^2+b^2)^{5/2}}\\
&\quad-\frac{(-3ab^2\cos(x)+6b(-2a^2+b^2)\cos(2x)+(4a^2-b^2)(2b+a\cos(3x)))\csc^3(x)}{12(a-b)^2(a+b)^2}
\end{aligned}$$

```
[In] Integrate[Cot[x]^4/(a + b*Cos[x]),x]
```

```
[Out] (-2*a^4*ArcTanh[((a - b)*Tan[x/2])/Sqrt[-a^2 + b^2]])/(-a^2 + b^2)^(5/2) -
((-3*a*b^2*Cos[x] + 6*b*(-2*a^2 + b^2)*Cos[2*x] + (4*a^2 - b^2)*(2*b + a*Co
s[3*x]))*Csc[x]^3)/(12*(a - b)^2*(a + b)^2)
```

Maple [A] (verified)

Time = 1.36 (sec) , antiderivative size = 127, normalized size of antiderivative = 0.92

method	result
default	$\frac{\frac{a\left(\tan^3\left(\frac{x}{2}\right)\right)}{3}-\frac{b\left(\tan^3\left(\frac{x}{2}\right)\right)}{3}-5a\tan\left(\frac{x}{2}\right)+3b\tan\left(\frac{x}{2}\right)}{8(a-b)^2}+\frac{2a^4\arctan\left(\frac{(a-b)\tan\left(\frac{x}{2}\right)}{\sqrt{(a-b)(a+b)}}\right)}{(a-b)^2(a+b)^2\sqrt{(a-b)(a+b)}}-\frac{1}{24(a+b)\tan\left(\frac{x}{2}\right)^3}-\frac{-5a-3b}{8(a+b)^2\tan\left(\frac{x}{2}\right)}$
risch	$-\frac{2i\left(6a^2b\,\mathrm{e}^{5ix}-3b^3\mathrm{e}^{5ix}-6a^3\mathrm{e}^{4ix}+3a\,b^2\mathrm{e}^{4ix}-8a^2b\,\mathrm{e}^{3ix}+2b^3\mathrm{e}^{3ix}+6a^3\mathrm{e}^{2ix}+6a^2b\,\mathrm{e}^{ix}-3b^3\mathrm{e}^{ix}-4a^3+a\,b^2\right)}{3(a^4-2a^2b^2+b^4)(\mathrm{e}^{2ix}-1)^3}-\frac{a^4\ln\left(\mathrm{e}^{ix}+\frac{ia^2-ib^2+a\sqrt{-}}{b\sqrt{-a^2+}}\right.}{\sqrt{-a^2+b^2}\,(a+b)^2(a-}$

```
[In] int(cot(x)^4/(a+cos(x)*b),x,method=_RETURNVERBOSE)
```

```
[Out] 1/8/(a-b)^2*(1/3*a*tan(1/2*x)^3-1/3*b*tan(1/2*x)^3-5*a*tan(1/2*x)+3*b*tan(1
/2*x))+2/(a-b)^2/(a+b)^2*a^4/((a-b)*(a+b))^(1/2)*arctan((a-b)*tan(1/2*x)/((
a-b)*(a+b))^(1/2))-1/24/(a+b)/tan(1/2*x)^3-1/8/(a+b)^2*(-5*a-3*b)/tan(1/2*x
)
```

Fricas [A] (verification not implemented)

none

Time = 0.33 (sec) , antiderivative size = 456, normalized size of antiderivative = 3.30

$$\int \frac{\cot^4(x)}{a+b\cos(x)}\,dx$$
$$=\left[-\frac{10\,a^4b-14\,a^2b^3+4\,b^5+2\left(4\,a^5-5\,a^3b^2+ab^4\right)\cos\left(x\right)^3-3\left(a^4\cos\left(x\right)^2-a^4\right)\sqrt{-a^2+b^2}\log\left(\frac{2\,ab\cos(x)+}{}\right.}{6\left(a^6-3\,a^4b^2+3\,a^2b^4-b^6-\left(a^6\right.\right.}\right.$$
$$-\frac{5\,a^4b-7\,a^2b^3+2\,b^5+\left(4\,a^5-5\,a^3b^2+ab^4\right)\cos\left(x\right)^3+3\left(a^4\cos\left(x\right)^2-a^4\right)\sqrt{a^2-b^2}\arctan\left(-\frac{a\cos(x)+b}{\sqrt{a^2-b^2}\sin(x)}\right.}{3\left(a^6-3\,a^4b^2+3\,a^2b^4-b^6-\left(a^6-3\,a^4b^2+3\,a^2b^4-b^6\right)\right.}$$

```
[In] integrate(cot(x)^4/(a+b*cos(x)),x, algorithm="fricas")
```

```
[Out] [-1/6*(10*a^4*b - 14*a^2*b^3 + 4*b^5 + 2*(4*a^5 - 5*a^3*b^2 + a*b^4)*cos(x)
^3 - 3*(a^4*cos(x)^2 - a^4)*sqrt(-a^2 + b^2)*log((2*a*b*cos(x) + (2*a^2 - b
^2)*cos(x)^2 + 2*sqrt(-a^2 + b^2)*(a*cos(x) + b)*sin(x) - a^2 + 2*b^2)/(b^2
*cos(x)^2 + 2*a*b*cos(x) + a^2))*sin(x) - 6*(2*a^4*b - 3*a^2*b^3 + b^5)*cos
(x)^2 - 6*(a^5 - a^3*b^2)*cos(x))/((a^6 - 3*a^4*b^2 + 3*a^2*b^4 - b^6 - (a^
6 - 3*a^4*b^2 + 3*a^2*b^4 - b^6)*cos(x)^2)*sin(x)), -1/3*(5*a^4*b - 7*a^2*b
^3 + 2*b^5 + (4*a^5 - 5*a^3*b^2 + a*b^4)*cos(x)^3 + 3*(a^4*cos(x)^2 - a^4)*
sqrt(a^2 - b^2)*arctan(-(a*cos(x) + b)/(sqrt(a^2 - b^2)*sin(x)))*sin(x) - 3
*(2*a^4*b - 3*a^2*b^3 + b^5)*cos(x)^2 - 3*(a^5 - a^3*b^2)*cos(x))/((a^6 - 3
*a^4*b^2 + 3*a^2*b^4 - b^6 - (a^6 - 3*a^4*b^2 + 3*a^2*b^4 - b^6)*cos(x)^2)*
sin(x))]
```

Sympy [F]

$$\int \frac{\cot^4(x)}{a+b\cos(x)}\,dx = \int \frac{\cot^4{(x)}}{a+b\cos{(x)}}\,dx$$

```
[In] integrate(cot(x)**4/(a+b*cos(x)),x)
```

```
[Out] Integral(cot(x)**4/(a + b*cos(x)), x)
```

Maxima [F(-2)]

Exception generated.

$$\int \frac{\cot^4(x)}{a+b\cos(x)}\,dx = \text{Exception raised: ValueError}$$

```
[In] integrate(cot(x)^4/(a+b*cos(x)),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*b^2-4*a^2>0)', see `assume?` for
more de
```

Giac [A] (verification not implemented)

none

Time = 0.30 (sec) , antiderivative size = 210, normalized size of antiderivative = 1.52

$$\int \frac{\cot^4(x)}{a+b\cos(x)}\,dx = -\frac{2\left(\pi\left\lfloor\frac{x}{2\pi}+\frac{1}{2}\right\rfloor\operatorname{sgn}(-2a+2b)+\arctan\left(-\frac{a\tan\left(\frac{1}{2}x\right)-b\tan\left(\frac{1}{2}x\right)}{\sqrt{a^2-b^2}}\right)\right)a^4}{(a^4-2a^2b^2+b^4)\sqrt{a^2-b^2}} + \frac{a^2\tan\left(\frac{1}{2}x\right)^3-2ab\tan\left(\frac{1}{2}x\right)^3+b^2\tan\left(\frac{1}{2}x\right)^3-15a^2\tan\left(\frac{1}{2}x\right)+24ab\tan\left(\frac{1}{2}x\right)-9b^2\tan\left(\frac{1}{2}x\right)}{24(a^3-3a^2b+3ab^2-b^3)} + \frac{15a\tan\left(\frac{1}{2}x\right)^2+9b\tan\left(\frac{1}{2}x\right)^2-a-b}{24(a^2+2ab+b^2)\tan\left(\frac{1}{2}x\right)^3}$$

```
[In] integrate(cot(x)^4/(a+b*cos(x)),x, algorithm="giac")
```

```
[Out] -2*(pi*floor(1/2*x/pi + 1/2)*sgn(-2*a + 2*b) + arctan(-(a*tan(1/2*x) - b*ta
n(1/2*x))/sqrt(a^2 - b^2)))*a^4/((a^4 - 2*a^2*b^2 + b^4)*sqrt(a^2 - b^2)) +
 1/24*(a^2*tan(1/2*x)^3 - 2*a*b*tan(1/2*x)^3 + b^2*tan(1/2*x)^3 - 15*a^2*ta
n(1/2*x) + 24*a*b*tan(1/2*x) - 9*b^2*tan(1/2*x))/(a^3 - 3*a^2*b + 3*a*b^2 -
 b^3) + 1/24*(15*a*tan(1/2*x)^2 + 9*b*tan(1/2*x)^2 - a - b)/((a^2 + 2*a*b +
 b^2)*tan(1/2*x)^3)
```

Mupad [B] (verification not implemented)

Time = 14.33 (sec) , antiderivative size = 183, normalized size of antiderivative = 1.33

$$\int \frac{\cot^4(x)}{a+b\cos(x)}\,dx = \frac{\tan\left(\frac{x}{2}\right)^3}{3\,(8\,a-8\,b)} - \tan\left(\frac{x}{2}\right)\,\left(\frac{4}{8\,a-8\,b}+\frac{8\,a+8\,b}{(8\,a-8\,b)^2}\right) - \frac{\frac{a^2-2\,a\,b+b^2}{3\,(a+b)}+\frac{\tan\left(\frac{x}{2}\right)^2\,(-5\,a^3+7\,a^2\,b+a\,b^2-3\,b^3)}{(a+b)^2}}{\tan\left(\frac{x}{2}\right)^3\,(8\,a^2-16\,a\,b+8\,b^2)} + \frac{2\,a^4\,\mathrm{atan}\left(\frac{\tan\left(\frac{x}{2}\right)\,(a^4-2\,a^2\,b^2+b^4)}{(a+b)^{5/2}\,(a-b)^{3/2}}\right)}{(a+b)^{5/2}\,(a-b)^{5/2}}$$

```
[In] int(cot(x)^4/(a + b*cos(x)),x)
```

```
[Out] tan(x/2)^3/(3*(8*a - 8*b)) - tan(x/2)*(4/(8*a - 8*b) + (8*a + 8*b)/(8*a - 8
*b)^2) - ((a^2 - 2*a*b + b^2)/(3*(a + b)) + (tan(x/2)^2*(a*b^2 + 7*a^2*b -
5*a^3 - 3*b^3))/(a + b)^2)/(tan(x/2)^3*(8*a^2 - 16*a*b + 8*b^2)) + (2*a^4*a
tan((tan(x/2)*(a^4 + b^4 - 2*a^2*b^2))/((a + b)^(5/2)*(a - b)^(3/2))))/((a
+ b)^(5/2)*(a - b)^(5/2))
```

3.18 $\int \frac{\cot(x)}{\sqrt{3-\cos(x)}}\,dx$

Optimal result

Integrand size = 13, antiderivative size = 44

$$\int \frac{\cot(x)}{\sqrt{3-\cos(x)}}\,dx = -\frac{1}{2}\operatorname{arctanh}\left(\frac{1}{2}\sqrt{3-\cos(x)}\right) - \frac{\operatorname{arctanh}\left(\frac{\sqrt{3-\cos(x)}}{\sqrt{2}}\right)}{\sqrt{2}}$$

```
[Out] -1/2*arctanh(1/2*(3-cos(x))^(1/2))-1/2*arctanh(1/2*(3-cos(x))^(1/2)*2^(1/2)
)*2^(1/2)
```

Rubi [A] (verified)

Time = 0.06 (sec) , antiderivative size = 44, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 4, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.308, Rules used = {2800, 841, 1180, 212}

$$\int \frac{\cot(x)}{\sqrt{3-\cos(x)}}\,dx = -\frac{1}{2}\operatorname{arctanh}\left(\frac{1}{2}\sqrt{3-\cos(x)}\right) - \frac{\operatorname{arctanh}\left(\frac{\sqrt{3-\cos(x)}}{\sqrt{2}}\right)}{\sqrt{2}}$$

```
[In] Int[Cot[x]/Sqrt[3 - Cos[x]],x]
```

```
[Out] -1/2*ArcTanh[Sqrt[3 - Cos[x]]/2] - ArcTanh[Sqrt[3 - Cos[x]]/Sqrt[2]]/Sqrt[2
]
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 841

```
Int[((f_.) + (g_.)*(x_))/(Sqrt[(d_.) + (e_.)*(x_)]*((a_) + (c_.)*(x_)^2)),
x_Symbol] :> Dist[2, Subst[Int[(e*f - d*g + g*x^2)/(c*d^2 + a*e^2 - 2*c*d*x
^2 + c*x^4), x], x, Sqrt[d + e*x]], x] /; FreeQ[{a, c, d, e, f, g}, x] && N
eQ[c*d^2 + a*e^2, 0]
```

Rule 1180

```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Dist[e/2 + (2*c*d - b*e)/(2*q), Int[1/(b/2
 - q/2 + c*x^2), x], x] + Dist[e/2 - (2*c*d - b*e)/(2*q), Int[1/(b/2 + q/2
+ c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && Ne
Q[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

Rule 2800

```
Int[((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m_.)*tan[(e_.) + (f_.)*(x_)]^(p
_.), x_Symbol] :> Dist[1/f, Subst[Int[(x^p*(a + x)^m)/(b^2 - x^2)^((p + 1)/
2), x], x, b*Sin[e + f*x]], x] /; FreeQ[{a, b, e, f, m}, x] && NeQ[a^2 - b^
2, 0] && IntegerQ[(p + 1)/2]
```

Rubi steps

$$
\begin{aligned}
\text{integral} &= -\text{Subst}\left(\int \frac{x}{\sqrt{3+x}\,(1-x^2)}\,dx, x, -\cos(x)\right)\\
&= -\left(2\text{Subst}\left(\int \frac{-3+x^2}{-8+6x^2-x^4}\,dx, x, \sqrt{3-\cos(x)}\right)\right)\\
&= -\text{Subst}\left(\int \frac{1}{2-x^2}\,dx, x, \sqrt{3-\cos(x)}\right) - \text{Subst}\left(\int \frac{1}{4-x^2}\,dx, x, \sqrt{3-\cos(x)}\right)\\
&= -\frac{1}{2}\text{arctanh}\left(\frac{1}{2}\sqrt{3-\cos(x)}\right) - \frac{\text{arctanh}\left(\frac{\sqrt{3-\cos(x)}}{\sqrt{2}}\right)}{\sqrt{2}}
\end{aligned}
$$

Mathematica [A] (verified)

Time = 0.08 (sec) , antiderivative size = 44, normalized size of antiderivative = 1.00

$$
\int \frac{\cot(x)}{\sqrt{3-\cos(x)}}\,dx = -\frac{1}{2}\text{arctanh}\left(\frac{1}{2}\sqrt{3-\cos(x)}\right) - \frac{\text{arctanh}\left(\frac{\sqrt{3-\cos(x)}}{\sqrt{2}}\right)}{\sqrt{2}}
$$

```
[In] Integrate[Cot[x]/Sqrt[3 - Cos[x]],x]
[Out] -1/2*ArcTanh[Sqrt[3 - Cos[x]]/2] - ArcTanh[Sqrt[3 - Cos[x]]/Sqrt[2]]/Sqrt[2
]
```

Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 80 vs. $2(33) = 66$.

Time = 1.68 (sec) , antiderivative size = 81, normalized size of antiderivative = 1.84

method	result	size
default	$-\frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{\left(2+\cos\left(\frac{x}{2}\right)\right)\sqrt{2}}{\sqrt{2\left(\sin^2\left(\frac{x}{2}\right)\right)+2}}\right)}{4}-\frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{-\sqrt{2}\,\cos\left(\frac{x}{2}\right)+2\sqrt{2}}{\sqrt{2\left(\sin^2\left(\frac{x}{2}\right)\right)+2}}\right)}{4}-\frac{\operatorname{arctanh}\left(\frac{2}{\sqrt{2\left(\sin^2\left(\frac{x}{2}\right)\right)+2}}\right)}{2}$	81

```
[In] int(cot(x)/(3-cos(x))^(1/2),x,method=_RETURNVERBOSE)
[Out] -1/4*2^(1/2)*arctanh(1/(2*sin(1/2*x)^2+2)^(1/2)*(2+cos(1/2*x))*2^(1/2))-1/4
*2^(1/2)*arctanh((-2^(1/2)*cos(1/2*x)+2*2^(1/2))/(2*sin(1/2*x)^2+2)^(1/2))-
1/2*arctanh(2/(2*sin(1/2*x)^2+2)^(1/2))
```

Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 77 vs. $2(33) = 66$.

Time = 0.29 (sec) , antiderivative size = 77, normalized size of antiderivative = 1.75

$$\begin{aligned}\int \frac{\cot(x)}{\sqrt{3-\cos(x)}}\,dx &= \frac{1}{8}\sqrt{2}\log\left(\frac{\cos(x)^2+4\left(\sqrt{2}\cos(x)-5\sqrt{2}\right)\sqrt{-\cos(x)+3}-18\cos(x)+49}{\cos(x)^2-2\cos(x)+1}\right)\\ &\quad+\frac{1}{4}\log\left(-\frac{4\sqrt{-\cos(x)+3}+\cos(x)-7}{\cos(x)+1}\right)\end{aligned}$$

```
[In] integrate(cot(x)/(3-cos(x))^(1/2),x, algorithm="fricas")
[Out] 1/8*sqrt(2)*log((cos(x)^2 + 4*(sqrt(2)*cos(x) - 5*sqrt(2))*sqrt(-cos(x) + 3
) - 18*cos(x) + 49)/(cos(x)^2 - 2*cos(x) + 1)) + 1/4*log(-(4*sqrt(-cos(x) +
 3) + cos(x) - 7)/(cos(x) + 1))
```

Sympy [F]

$$\int \frac{\cot(x)}{\sqrt{3-\cos(x)}}\,dx = \int \frac{\cot(x)}{\sqrt{3-\cos(x)}}\,dx$$

```
[In] integrate(cot(x)/(3-cos(x))**(1/2),x)
[Out] Integral(cot(x)/sqrt(3 - cos(x)), x)
```

Maxima [A] (verification not implemented)

none

Time = 0.30 (sec) , antiderivative size = 63, normalized size of antiderivative = 1.43

$$\int \frac{\cot(x)}{\sqrt{3-\cos(x)}}\,dx = \frac{1}{4}\sqrt{2}\log\left(-\frac{\sqrt{2}-\sqrt{-\cos(x)+3}}{\sqrt{2}+\sqrt{-\cos(x)+3}}\right) - \frac{1}{4}\log\left(\sqrt{-\cos(x)+3}+2\right) + \frac{1}{4}\log\left(\sqrt{-\cos(x)+3}-2\right)$$

```
[In] integrate(cot(x)/(3-cos(x))^(1/2),x, algorithm="maxima")
[Out] 1/4*sqrt(2)*log(-(sqrt(2) - sqrt(-cos(x) + 3))/(sqrt(2) + sqrt(-cos(x) + 3)
)) - 1/4*log(sqrt(-cos(x) + 3) + 2) + 1/4*log(sqrt(-cos(x) + 3) - 2)
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 68 vs. 2(33) = 66.

Time = 0.31 (sec) , antiderivative size = 68, normalized size of antiderivative = 1.55

$$\int \frac{\cot(x)}{\sqrt{3-\cos(x)}}\,dx = \frac{1}{4}\sqrt{2}\log\left(\frac{\left|-2\sqrt{2}+2\sqrt{-\cos(x)+3}\right|}{2\left(\sqrt{2}+\sqrt{-\cos(x)+3}\right)}\right) - \frac{1}{4}\log\left(\sqrt{-\cos(x)+3}+2\right) + \frac{1}{4}\log\left(-\sqrt{-\cos(x)+3}+2\right)$$

```
[In] integrate(cot(x)/(3-cos(x))^(1/2),x, algorithm="giac")
[Out] 1/4*sqrt(2)*log(1/2*abs(-2*sqrt(2) + 2*sqrt(-cos(x) + 3))/(sqrt(2) + sqrt(-
cos(x) + 3))) - 1/4*log(sqrt(-cos(x) + 3) + 2) + 1/4*log(-sqrt(-cos(x) + 3)
 + 2)
```

Mupad [F(-1)]

Timed out.

$$\int \frac{\cot(x)}{\sqrt{3-\cos(x)}}\,dx = \int \frac{\cot(x)}{\sqrt{3-\cos(x)}}\,dx$$

```
[In] int(cot(x)/(3 - cos(x))^(1/2),x)
[Out] int(cot(x)/(3 - cos(x))^(1/2), x)
```

3.19 $\int \sqrt{a+b\cos(x)}\tan(x)\,dx$

Optimal result

Integrand size = 13, antiderivative size = 37

$$\int \sqrt{a+b\cos(x)}\tan(x)\,dx = 2\sqrt{a}\operatorname{arctanh}\left(\frac{\sqrt{a+b\cos(x)}}{\sqrt{a}}\right) - 2\sqrt{a+b\cos(x)}$$

```
[Out] 2*arctanh((a+b*cos(x))^(1/2)/a^(1/2))*a^(1/2)-2*(a+b*cos(x))^(1/2)
```

Rubi [A] (verified)

Time = 0.06 (sec) , antiderivative size = 37, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4, $\frac{\text{number of rules}}{\text{integrand size}} = 0.308$, Rules used = {2800, 52, 65, 213}

$$\int \sqrt{a+b\cos(x)}\tan(x)\,dx = 2\sqrt{a}\operatorname{arctanh}\left(\frac{\sqrt{a+b\cos(x)}}{\sqrt{a}}\right) - 2\sqrt{a+b\cos(x)}$$

```
[In] Int[Sqrt[a + b*Cos[x]]*Tan[x],x]
```

```
[Out] 2*Sqrt[a]*ArcTanh[Sqrt[a + b*Cos[x]]/Sqrt[a]] - 2*Sqrt[a + b*Cos[x]]
```

Rule 52

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[
(a + b*x)^(m + 1)*((c + d*x)^n/(b*(m + n + 1))), x] + Dist[n*((b*c - a*d)/(
b*(m + n + 1))), Int[(a + b*x)^m*(c + d*x)^(n - 1), x], x] /; FreeQ[{a, b,
c, d}, x] && NeQ[b*c - a*d, 0] && GtQ[n, 0] && NeQ[m + n + 1, 0] &&  !(IGtQ
[m, 0] && ( !IntegerQ[n] || (GtQ[m, 0] && LtQ[m - n, 0]))) &&  !ILtQ[m + n
+ 2, 0] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 65

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Dist[p/b, Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +
d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[b*c - a*d, 0] && LtQ[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Den
ominator[m]] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 213

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[b, 2])^(-
1))*ArcTanh[Rt[b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] &&
 (LtQ[a, 0] || GtQ[b, 0])
```

Rule 2800

```
Int[((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m_.)*tan[(e_.) + (f_.)*(x_)]^(p
_.), x_Symbol] :> Dist[1/f, Subst[Int[(x^p*(a + x)^m)/(b^2 - x^2)^((p + 1)/
2), x], x, b*Sin[e + f*x]], x] /; FreeQ[{a, b, e, f, m}, x] && NeQ[a^2 - b^
2, 0] && IntegerQ[(p + 1)/2]
```

Rubi steps

$$\begin{aligned}\text{integral} &= -\text{Subst}\left(\int \frac{\sqrt{a+x}}{x}\,dx, x, b\cos(x)\right)\\ &= -2\sqrt{a+b\cos(x)} - a\text{Subst}\left(\int \frac{1}{x\sqrt{a+x}}\,dx, x, b\cos(x)\right)\\ &= -2\sqrt{a+b\cos(x)} - (2a)\text{Subst}\left(\int \frac{1}{-a+x^2}\,dx, x, \sqrt{a+b\cos(x)}\right)\\ &= 2\sqrt{a}\,\text{arctanh}\left(\frac{\sqrt{a+b\cos(x)}}{\sqrt{a}}\right) - 2\sqrt{a+b\cos(x)}\end{aligned}$$

Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 37, normalized size of antiderivative = 1.00

$$\int \sqrt{a+b\cos(x)}\tan(x)\,dx = 2\sqrt{a}\,\text{arctanh}\left(\frac{\sqrt{a+b\cos(x)}}{\sqrt{a}}\right) - 2\sqrt{a+b\cos(x)}$$

```
[In] Integrate[Sqrt[a + b*Cos[x]]*Tan[x],x]
[Out] 2*Sqrt[a]*ArcTanh[Sqrt[a + b*Cos[x]]/Sqrt[a]] - 2*Sqrt[a + b*Cos[x]]
```

Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 122 vs. 2(29) = 58.

Time = 2.23 (sec) , antiderivative size = 123, normalized size of antiderivative = 3.32

method	result
default	$-2\sqrt{-2b\left(\sin^2\left(\frac{x}{2}\right)\right)+a+b}+\sqrt{a}\,\ln\left(\frac{4\cos\left(\frac{x}{2}\right)b\sqrt{2}+4\sqrt{a}\,\sqrt{-2b\left(\sin^2\left(\frac{x}{2}\right)\right)+a+b}+4a-4b}{2\cos\left(\frac{x}{2}\right)-\sqrt{2}}\right)+\sqrt{a}\,\ln\left(-\frac{4\left(\cos\right.}{}\right.$

```
[In] int((a+cos(x)*b)^(1/2)*tan(x),x,method=_RETURNVERBOSE)
```

```
[Out] -2*(-2*b*sin(1/2*x)^2+a+b)^(1/2)+a^(1/2)*ln(4/(2*cos(1/2*x)-2^(1/2))*(cos(1
/2*x)*b*2^(1/2)+a^(1/2)*(-2*b*sin(1/2*x)^2+a+b)^(1/2)+a-b))+a^(1/2)*ln(-4/(
2*cos(1/2*x)+2^(1/2))*(cos(1/2*x)*b*2^(1/2)-a^(1/2)*(-2*b*sin(1/2*x)^2+a+b)
^(1/2)-a+b))
```

Fricas [A] (verification not implemented)

none

Time = 0.40 (sec) , antiderivative size = 109, normalized size of antiderivative = 2.95

$$\int \sqrt{a+b\cos(x)}\tan(x)\,dx = \left[\frac{1}{2}\sqrt{a}\log\left(-\frac{b^2\cos(x)^2+8ab\cos(x)+4(b\cos(x)+2a)\sqrt{b\cos(x)+a}\sqrt{a}+8a^2}{\cos(x)^2}\right) - 2\sqrt{b\cos(x)+a}, -\sqrt{-a}\arctan\left(\frac{2\sqrt{b\cos(x)+a}\sqrt{-a}}{b\cos(x)+2a}\right) - 2\sqrt{b\cos(x)+a}\right]$$

```
[In] integrate((a+b*cos(x))^(1/2)*tan(x),x, algorithm="fricas")
```

```
[Out] [1/2*sqrt(a)*log(-(b^2*cos(x)^2 + 8*a*b*cos(x) + 4*(b*cos(x) + 2*a)*sqrt(b*
cos(x) + a)*sqrt(a) + 8*a^2)/cos(x)^2) - 2*sqrt(b*cos(x) + a), -sqrt(-a)*ar
ctan(2*sqrt(b*cos(x) + a)*sqrt(-a)/(b*cos(x) + 2*a)) - 2*sqrt(b*cos(x) + a)
]
```

Sympy [F]

$$\int \sqrt{a+b\cos(x)}\tan(x)\,dx = \int \sqrt{a+b\cos{(x)}}\tan{(x)}\,dx$$

```
[In] integrate((a+b*cos(x))**(1/2)*tan(x),x)
[Out] Integral(sqrt(a + b*cos(x))*tan(x), x)
```

Maxima [A] (verification not implemented)

none

Time = 0.29 (sec) , antiderivative size = 46, normalized size of antiderivative = 1.24

$$\int \sqrt{a+b\cos(x)}\tan(x)\,dx = -\sqrt{a}\log\left(\frac{\sqrt{b\cos{(x)}+a}-\sqrt{a}}{\sqrt{b\cos{(x)}+a}+\sqrt{a}}\right) - 2\sqrt{b\cos{(x)}+a}$$

```
[In] integrate((a+b*cos(x))^(1/2)*tan(x),x, algorithm="maxima")
[Out] -sqrt(a)*log((sqrt(b*cos(x) + a) - sqrt(a))/(sqrt(b*cos(x) + a) + sqrt(a)))
 - 2*sqrt(b*cos(x) + a)
```

Giac [A] (verification not implemented)

none

Time = 0.31 (sec) , antiderivative size = 34, normalized size of antiderivative = 0.92

$$\int \sqrt{a+b\cos(x)}\tan(x)\,dx = -\frac{2\,a\arctan\left(\frac{\sqrt{b\cos(x)+a}}{\sqrt{-a}}\right)}{\sqrt{-a}} - 2\sqrt{b\cos{(x)}+a}$$

```
[In] integrate((a+b*cos(x))^(1/2)*tan(x),x, algorithm="giac")
[Out] -2*a*arctan(sqrt(b*cos(x) + a)/sqrt(-a))/sqrt(-a) - 2*sqrt(b*cos(x) + a)
```

Mupad [F(-1)]

Timed out.

$$\int \sqrt{a+b\cos(x)}\tan(x)\,dx = \int \tan(x)\ \sqrt{a+b\ \cos(x)}\,dx$$

```
[In] int(tan(x)*(a + b*cos(x))^(1/2),x)
[Out] int(tan(x)*(a + b*cos(x))^(1/2), x)
```

3.20 $\int \frac{\tan(x)}{\sqrt{a+b\cos(x)}}\,dx$

Optimal result

Integrand size = 13, antiderivative size = 24

$$\int \frac{\tan(x)}{\sqrt{a+b\cos(x)}}\,dx=\frac{2\operatorname{arctanh}\left(\frac{\sqrt{a+b\cos(x)}}{\sqrt{a}}\right)}{\sqrt{a}}$$

```
[Out] 2*arctanh((a+b*cos(x))^(1/2)/a^(1/2))/a^(1/2)
```

Rubi [A] (verified)

Time = 0.06 (sec) , antiderivative size = 24, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.231, Rules used = {2800, 65, 213}

$$\int \frac{\tan(x)}{\sqrt{a+b\cos(x)}}\,dx=\frac{2\operatorname{arctanh}\left(\frac{\sqrt{a+b\cos(x)}}{\sqrt{a}}\right)}{\sqrt{a}}$$

```
[In] Int[Tan[x]/Sqrt[a + b*Cos[x]],x]
```

```
[Out] (2*ArcTanh[Sqrt[a + b*Cos[x]]/Sqrt[a]])/Sqrt[a]
```

Rule 65

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Dist[p/b, Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +
d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[b*c - a*d, 0] && LtQ[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Den
ominator[m]] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 213

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[b, 2])^(-
1))*ArcTanh[Rt[b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] &&
 (LtQ[a, 0] || GtQ[b, 0])
```

Rule 2800

```
Int[((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m_.)*tan[(e_.) + (f_.)*(x_)]^(p
_.), x_Symbol] :> Dist[1/f, Subst[Int[(x^p*(a + x)^m)/(b^2 - x^2)^((p + 1)/
2), x], x, b*Sin[e + f*x]], x] /; FreeQ[{a, b, e, f, m}, x] && NeQ[a^2 - b^
2, 0] && IntegerQ[(p + 1)/2]
```

Rubi steps

$$\begin{aligned}\text{integral} &= -\text{Subst}\left(\int \frac{1}{x\sqrt{a+x}}\,dx, x, b\cos(x)\right)\\ &= -\left(2\text{Subst}\left(\int \frac{1}{-a+x^2}\,dx, x, \sqrt{a+b\cos(x)}\right)\right)\\ &= \frac{2\text{arctanh}\left(\frac{\sqrt{a+b\cos(x)}}{\sqrt{a}}\right)}{\sqrt{a}}\end{aligned}$$

Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 24, normalized size of antiderivative = 1.00

$$\int \frac{\tan(x)}{\sqrt{a+b\cos(x)}}\,dx = \frac{2\text{arctanh}\left(\frac{\sqrt{a+b\cos(x)}}{\sqrt{a}}\right)}{\sqrt{a}}$$

```
[In] Integrate[Tan[x]/Sqrt[a + b*Cos[x]],x]
[Out] (2*ArcTanh[Sqrt[a + b*Cos[x]]/Sqrt[a]])/Sqrt[a]
```

Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 102 vs. 2(18) = 36.

Time = 1.71 (sec) , antiderivative size = 103, normalized size of antiderivative = 4.29

method	result	size
default	$\frac{\ln\left(\frac{4\cos\left(\frac{x}{2}\right)b\sqrt{2}+4\sqrt{a}\,\sqrt{-2b\left(\sin^2\left(\frac{x}{2}\right)\right)+a+b}+4a-4b}{2\cos\left(\frac{x}{2}\right)-\sqrt{2}}\right)+\ln\left(-\frac{4\left(\cos\left(\frac{x}{2}\right)b\sqrt{2}-\sqrt{a}\,\sqrt{-2b\left(\sin^2\left(\frac{x}{2}\right)\right)+a+b}-a+b\right)}{2\cos\left(\frac{x}{2}\right)+\sqrt{2}}\right)}{\sqrt{a}}$	103

```
[In] int(tan(x)/(a+cos(x)*b)^(1/2),x,method=_RETURNVERBOSE)
```

```
[Out] (ln(4/(2*cos(1/2*x)-2^(1/2))*(cos(1/2*x)*b*2^(1/2)+a^(1/2)*(-2*b*sin(1/2*x)
^2+a+b)^(1/2)+a-b))+ln(-4/(2*cos(1/2*x)+2^(1/2))*(cos(1/2*x)*b*2^(1/2)-a^(1
/2)*(-2*b*sin(1/2*x)^2+a+b)^(1/2)-a+b)))/a^(1/2)
```

Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 45 vs. $2(18) = 36$.

Time = 0.29 (sec) , antiderivative size = 98, normalized size of antiderivative = 4.08

$$\int \frac{\tan(x)}{\sqrt{a+b\cos(x)}}\,dx = \left[\frac{\log\left(\frac{b^2\cos(x)^2+8\,ab\cos(x)+4\,(b\cos(x)+2\,a)\sqrt{b\cos(x)+a}\sqrt{a}+8\,a^2}{\cos(x)^2}\right)}{2\,\sqrt{a}}, -\frac{\sqrt{-a}\arctan\left(\frac{(b\cos(x)+2\,a)\sqrt{b\cos(x)+a}\sqrt{-a}}{2\,(ab\cos(x)+a^2)}\right)}{a}\right]$$

```
[In] integrate(tan(x)/(a+b*cos(x))^(1/2),x, algorithm="fricas")
```

```
[Out] [1/2*log((b^2*cos(x)^2 + 8*a*b*cos(x) + 4*(b*cos(x) + 2*a)*sqrt(b*cos(x) +
a)*sqrt(a) + 8*a^2)/cos(x)^2)/sqrt(a), -sqrt(-a)*arctan(1/2*(b*cos(x) + 2*a
)*sqrt(b*cos(x) + a)*sqrt(-a)/(a*b*cos(x) + a^2))/a]
```

Sympy [F]

$$\int \frac{\tan(x)}{\sqrt{a+b\cos(x)}}\,dx = \int \frac{\tan{(x)}}{\sqrt{a+b\cos{(x)}}}\,dx$$

```
[In] integrate(tan(x)/(a+b*cos(x))**(1/2),x)
```

```
[Out] Integral(tan(x)/sqrt(a + b*cos(x)), x)
```

Maxima [A] (verification not implemented)

none

Time = 0.30 (sec) , antiderivative size = 35, normalized size of antiderivative = 1.46

$$\int \frac{\tan(x)}{\sqrt{a+b\cos(x)}}\,dx = -\frac{\log\left(\frac{\sqrt{b\cos(x)+a}-\sqrt{a}}{\sqrt{b\cos(x)+a}+\sqrt{a}}\right)}{\sqrt{a}}$$

```
[In] integrate(tan(x)/(a+b*cos(x))^(1/2),x, algorithm="maxima")
```

```
[Out] -log((sqrt(b*cos(x) + a) - sqrt(a))/(sqrt(b*cos(x) + a) + sqrt(a)))/sqrt(a)
```

Giac [A] (verification not implemented)

none

Time = 0.31 (sec) , antiderivative size = 22, normalized size of antiderivative = 0.92

$$\int \frac{\tan(x)}{\sqrt{a+b\cos(x)}}\,dx = -\frac{2\arctan\left(\frac{\sqrt{b\cos(x)+a}}{\sqrt{-a}}\right)}{\sqrt{-a}}$$

```
[In] integrate(tan(x)/(a+b*cos(x))^(1/2),x, algorithm="giac")
[Out] -2*arctan(sqrt(b*cos(x) + a)/sqrt(-a))/sqrt(-a)
```

Mupad [F(-1)]

Timed out.

$$\int \frac{\tan(x)}{\sqrt{a+b\cos(x)}}\,dx = \int \frac{\tan(x)}{\sqrt{a+b\,\cos(x)}}\,dx$$

```
[In] int(tan(x)/(a + b*cos(x))^(1/2),x)
[Out] int(tan(x)/(a + b*cos(x))^(1/2), x)
```

3.21 $\int \frac{\sqrt{e \tan(c+dx)}}{a+b\cos(c+dx)}\,dx$

Optimal result

Integrand size = 25, antiderivative size = 204

$$\int \frac{\sqrt{e\tan(c+dx)}}{a+b\cos(c+dx)}\,dx = -\frac{2\sqrt{2}\sqrt{\cos(c+dx)}\,\text{EllipticPi}\left(-\frac{\sqrt{-a+b}}{\sqrt{a+b}},\arcsin\left(\frac{\sqrt{\sin(c+dx)}}{\sqrt{1+\cos(c+dx)}}\right),-1\right)\sqrt{e\tan(c+dx)}}{\sqrt{-a+b}\sqrt{a+b}d\sqrt{\sin(c+dx)}} + \frac{2\sqrt{2}\sqrt{\cos(c+dx)}\,\text{EllipticPi}\left(\frac{\sqrt{-a+b}}{\sqrt{a+b}},\arcsin\left(\frac{\sqrt{\sin(c+dx)}}{\sqrt{1+\cos(c+dx)}}\right),-1\right)\sqrt{e\tan(c+dx)}}{\sqrt{-a+b}\sqrt{a+b}d\sqrt{\sin(c+dx)}}$$

```
[Out] -2*EllipticPi(sin(d*x+c)^(1/2)/(1+cos(d*x+c))^(1/2),-(-a+b)^(1/2)/(a+b)^(1/
2),I)*2^(1/2)*cos(d*x+c)^(1/2)*(e*tan(d*x+c))^(1/2)/d/(-a+b)^(1/2)/(a+b)^(1
/2)/sin(d*x+c)^(1/2)+2*EllipticPi(sin(d*x+c)^(1/2)/(1+cos(d*x+c))^(1/2),(-a
+b)^(1/2)/(a+b)^(1/2),I)*2^(1/2)*cos(d*x+c)^(1/2)*(e*tan(d*x+c))^(1/2)/d/(-
a+b)^(1/2)/(a+b)^(1/2)/sin(d*x+c)^(1/2)
```

Rubi [A] (verified)

Time = 0.63 (sec) , antiderivative size = 204, normalized size of antiderivative = 1.00, number of steps used = 9, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.280, Rules used

= {2812, 2809, 2985, 2984, 504, 1227, 551}

$$\int \frac{\sqrt{e\tan(c+dx)}}{a+b\cos(c+dx)}\,dx$$
$$= \frac{2\sqrt{2}\sqrt{\cos(c+dx)}\sqrt{e\tan(c+dx)}\,\text{EllipticPi}\left(\frac{\sqrt{b-a}}{\sqrt{a+b}},\arcsin\left(\frac{\sqrt{\sin(c+dx)}}{\sqrt{\cos(c+dx)+1}}\right),-1\right)}{d\sqrt{b-a}\sqrt{a+b}\sqrt{\sin(c+dx)}}$$
$$- \frac{2\sqrt{2}\sqrt{\cos(c+dx)}\sqrt{e\tan(c+dx)}\,\text{EllipticPi}\left(-\frac{\sqrt{b-a}}{\sqrt{a+b}},\arcsin\left(\frac{\sqrt{\sin(c+dx)}}{\sqrt{\cos(c+dx)+1}}\right),-1\right)}{d\sqrt{b-a}\sqrt{a+b}\sqrt{\sin(c+dx)}}$$

```
[In] Int[Sqrt[e*Tan[c + d*x]]/(a + b*Cos[c + d*x]),x]
```

```
[Out] (-2*Sqrt[2]*Sqrt[Cos[c + d*x]]*EllipticPi[-(Sqrt[-a + b]/Sqrt[a + b]), ArcS
in[Sqrt[Sin[c + d*x]]/Sqrt[1 + Cos[c + d*x]]], -1]*Sqrt[e*Tan[c + d*x]])/(S
qrt[-a + b]*Sqrt[a + b]*d*Sqrt[Sin[c + d*x]]) + (2*Sqrt[2]*Sqrt[Cos[c + d*x
]]*EllipticPi[Sqrt[-a + b]/Sqrt[a + b], ArcSin[Sqrt[Sin[c + d*x]]/Sqrt[1 +
Cos[c + d*x]]], -1]*Sqrt[e*Tan[c + d*x]])/(Sqrt[-a + b]*Sqrt[a + b]*d*Sqrt[
Sin[c + d*x]])
```

Rule 504

```
Int[(x_)^2/(((a_) + (b_.)*(x_)^4)*Sqrt[(c_) + (d_.)*(x_)^4]), x_Symbol] :>
With[{r = Numerator[Rt[-a/b, 2]], s = Denominator[Rt[-a/b, 2]]}, Dist[s/(2*
b), Int[1/((r + s*x^2)*Sqrt[c + d*x^4]), x], x] - Dist[s/(2*b), Int[1/((r -
 s*x^2)*Sqrt[c + d*x^4]), x], x]] /; FreeQ[{a, b, c, d}, x] && NeQ[b*c - a*
d, 0]
```

Rule 551

```
Int[1/(((a_) + (b_.)*(x_)^2)*Sqrt[(c_) + (d_.)*(x_)^2]*Sqrt[(e_) + (f_.)*(x
_)^2]), x_Symbol] :> Simp[(1/(a*Sqrt[c]*Sqrt[e]*Rt[-d/c, 2]))*EllipticPi[b*
(c/(a*d)), ArcSin[Rt[-d/c, 2]*x], c*(f/(d*e))], x] /; FreeQ[{a, b, c, d, e,
 f}, x] &&  !GtQ[d/c, 0] && GtQ[c, 0] && GtQ[e, 0] &&  !( !GtQ[f/e, 0] && S
implerSqrtQ[-f/e, -d/c])
```

Rule 1227

```
Int[1/(((d_) + (e_.)*(x_)^2)*Sqrt[(a_) + (c_.)*(x_)^4]), x_Symbol] :> With[
{q = Rt[(-a)*c, 2]}, Dist[Sqrt[-c], Int[1/((d + e*x^2)*Sqrt[q + c*x^2]*Sqrt
[q - c*x^2]), x], x]] /; FreeQ[{a, c, d, e}, x] && GtQ[a, 0] && LtQ[c, 0]
```

Rule 2809

```
Int[1/(((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])*Sqrt[(g_)*tan[(e_.) + (f_.)*(
x_)]]), x_Symbol] :> Dist[Sqrt[Sin[e + f*x]]/(Sqrt[Cos[e + f*x]]*Sqrt[g*Tan
[e + f*x]]), Int[Sqrt[Cos[e + f*x]]/(Sqrt[Sin[e + f*x]]*(a + b*Sin[e + f*x]
```

```
)), x], x] /; FreeQ[{a, b, e, f, g}, x] && NeQ[a^2 - b^2, 0]
```

Rule 2812

```
Int[(cot[(e_.) + (f_.)*(x_)]*(g_.))^(p_)*((a_) + (b_.)*sin[(e_.) + (f_.)*(x
_)])^(m_.), x_Symbol] :> Dist[g^(2*IntPart[p])*(g*Cot[e + f*x])^FracPart[p]
*(g*Tan[e + f*x])^FracPart[p], Int[(a + b*Sin[e + f*x])^m/(g*Tan[e + f*x])^
p, x], x] /; FreeQ[{a, b, e, f, g, m, p}, x] &&  !IntegerQ[p]
```

Rule 2984

```
Int[Sqrt[cos[(e_.) + (f_.)*(x_)]*(g_.)]/(Sqrt[sin[(e_.) + (f_.)*(x_)]]*((a_
) + (b_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Dist[-4*Sqrt[2]*(g/f), Su
bst[Int[x^2/(((a + b)*g^2 + (a - b)*x^4)*Sqrt[1 - x^4/g^2]), x], x, Sqrt[g*
Cos[e + f*x]]/Sqrt[1 + Sin[e + f*x]]], x] /; FreeQ[{a, b, e, f, g}, x] && N
eQ[a^2 - b^2, 0]
```

Rule 2985

```
Int[Sqrt[cos[(e_.) + (f_.)*(x_)]*(g_.)]/(Sqrt[(d_)*sin[(e_.) + (f_.)*(x_)]]
*((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Dist[Sqrt[Sin[e + f*
x]]/Sqrt[d*Sin[e + f*x]], Int[Sqrt[g*Cos[e + f*x]]/(Sqrt[Sin[e + f*x]]*(a +
 b*Sin[e + f*x])), x], x] /; FreeQ[{a, b, d, e, f, g}, x] && NeQ[a^2 - b^2,
 0]
```

Rubi steps

$$\text{integral} = \left(\sqrt{e\cot(c+dx)}\sqrt{e\tan(c+dx)}\right)\int \frac{1}{(a+b\cos(c+dx))\sqrt{e\cot(c+dx)}}\,dx$$

$$= \frac{\left(\sqrt{-\cos(c+dx)}\sqrt{e\tan(c+dx)}\right)\int \frac{\sqrt{\sin(c+dx)}}{\sqrt{-\cos(c+dx)}(a+b\cos(c+dx))}\,dx}{\sqrt{\sin(c+dx)}}$$

$$= \frac{\left(\sqrt{\cos(c+dx)}\sqrt{e\tan(c+dx)}\right)\int \frac{\sqrt{\sin(c+dx)}}{\sqrt{\cos(c+dx)}(a+b\cos(c+dx))}\,dx}{\sqrt{\sin(c+dx)}}$$

$$= \frac{\left(4\sqrt{2}\sqrt{\cos(c+dx)}\sqrt{e\tan(c+dx)}\right)\text{Subst}\left(\int \frac{x^2}{\sqrt{1-x^4}(a+b+(a-b)x^4)}\,dx, x, \frac{\sqrt{\sin(c+dx)}}{\sqrt{1+\cos(c+dx)}}\right)}{d\sqrt{\sin(c+dx)}}$$

$$= \frac{\left(2\sqrt{2}\sqrt{\cos(c+dx)}\sqrt{e\tan(c+dx)}\right)\text{Subst}\left(\int \frac{1}{\left(\sqrt{a+b}-\sqrt{-a+b}x^2\right)\sqrt{1-x^4}}\,dx, x, \frac{\sqrt{\sin(c+dx)}}{\sqrt{1+\cos(c+dx)}}\right)}{\sqrt{-a+b}d\sqrt{\sin(c+dx)}}$$

$$- \frac{\left(2\sqrt{2}\sqrt{\cos(c+dx)}\sqrt{e\tan(c+dx)}\right)\text{Subst}\left(\int \frac{1}{\left(\sqrt{a+b}+\sqrt{-a+b}x^2\right)\sqrt{1-x^4}}\,dx, x, \frac{\sqrt{\sin(c+dx)}}{\sqrt{1+\cos(c+dx)}}\right)}{\sqrt{-a+b}d\sqrt{\sin(c+dx)}}$$

$$= \frac{\left(2\sqrt{2}\sqrt{\cos(c+dx)}\sqrt{e\tan(c+dx)}\right)\text{Subst}\left(\int \frac{1}{\sqrt{1-x^2}\sqrt{1+x^2}\left(\sqrt{a+b}-\sqrt{-a+b}x^2\right)}\,dx, x, \frac{\sqrt{\sin(c+dx)}}{\sqrt{1+\cos(c+dx)}}\right)}{\sqrt{-a+b}d\sqrt{\sin(c+dx)}} - \frac{\left(2\sqrt{2}\sqrt{\cos(c+dx)}\sqrt{e\tan(c+dx)}\right)\text{Subst}\left(\int \frac{1}{\sqrt{1-x^2}\sqrt{1+x^2}\left(\sqrt{a+b}+\sqrt{-a+b}x^2\right)}\,dx, x, \frac{\sqrt{\sin(c+dx)}}{\sqrt{1+\cos(c+dx)}}\right)}{\sqrt{-a+b}d\sqrt{\sin(c+dx)}}$$

$$= -\frac{2\sqrt{2}\sqrt{\cos(c+dx)}\,\text{EllipticPi}\left(-\frac{\sqrt{-a+b}}{\sqrt{a+b}}, \arcsin\left(\frac{\sqrt{\sin(c+dx)}}{\sqrt{1+\cos(c+dx)}}\right), -1\right)\sqrt{e\tan(c+dx)}}{\sqrt{-a+b}\sqrt{a+b}d\sqrt{\sin(c+dx)}} + \frac{2\sqrt{2}\sqrt{\cos(c+dx)}\,\text{EllipticPi}\left(\frac{\sqrt{-a+b}}{\sqrt{a+b}}, \arcsin\left(\frac{\sqrt{\sin(c+dx)}}{\sqrt{1+\cos(c+dx)}}\right), -1\right)\sqrt{e\tan(c+dx)}}{\sqrt{-a+b}\sqrt{a+b}d\sqrt{\sin(c+dx)}}$$

Mathematica [C] (warning: unable to verify)

Result contains higher order function than in optimal. Order 6 vs. order 4 in optimal.

Time = 14.92 (sec) , antiderivative size = 363, normalized size of antiderivative = 1.78

$$\int \frac{\sqrt{e\tan(c+dx)}}{a+b\cos(c+dx)}\,dx$$

$$= \frac{2\left(b+a\sqrt{\sec^2(c+dx)}\right)\sqrt{e\tan(c+dx)}\left(\frac{-2\arctan\left(1-\frac{\sqrt{2}\sqrt{a}\sqrt{\tan(c+dx)}}{\sqrt[4]{a^2-b^2}}\right)+2\arctan\left(1+\frac{\sqrt{2}\sqrt{a}\sqrt{\tan(c+dx)}}{\sqrt[4]{a^2-b^2}}\right)+\log\left(\sqrt{a^2-b^2}-\sqrt{\right.}{}}{}\right.}{d(a+b\cos}$$

```
[In] Integrate[Sqrt[e*Tan[c + d*x]]/(a + b*Cos[c + d*x]),x]
```

```
[Out] (2*(b + a*Sqrt[Sec[c + d*x]^2])*Sqrt[e*Tan[c + d*x]]*((-2*ArcTan[1 - (Sqrt[
2]*Sqrt[a]*Sqrt[Tan[c + d*x]])/(a^2 - b^2)^(1/4)] + 2*ArcTan[1 + (Sqrt[2]*S
qrt[a]*Sqrt[Tan[c + d*x]])/(a^2 - b^2)^(1/4)] + Log[Sqrt[a^2 - b^2] - Sqrt[
2]*Sqrt[a]*(a^2 - b^2)^(1/4)*Sqrt[Tan[c + d*x]] + a*Tan[c + d*x]] - Log[Sqr
t[a^2 - b^2] + Sqrt[2]*Sqrt[a]*(a^2 - b^2)^(1/4)*Sqrt[Tan[c + d*x]] + a*Tan
[c + d*x]])/(4*Sqrt[2]*Sqrt[a]*(a^2 - b^2)^(1/4)) + (b*AppellF1[3/4, 1/2, 1
, 7/4, -Tan[c + d*x]^2, -((a^2*Tan[c + d*x]^2)/(a^2 - b^2))]*Tan[c + d*x]^(
3/2))/(3*(-a^2 + b^2))))/(d*(a + b*Cos[c + d*x])*Sqrt[Sec[c + d*x]^2]*Sqrt[
Tan[c + d*x]])
```

Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 453 vs. $2(164) = 328$.

Time = 3.76 (sec) , antiderivative size = 454, normalized size of antiderivative = 2.23

method	result
default	$-\frac{\left(\sqrt{-a^2+b^2}\,\Pi\left(\sqrt{-\cot(dx+c)+\csc(dx+c)+1},\frac{a-b}{a-b+\sqrt{-(a-b)(a+b)}},\frac{\sqrt{2}}{2}\right)-\sqrt{-a^2+b^2}\,\Pi\left(\sqrt{-\cot(dx+c)+\csc(dx+c)+1},-\frac{a-}{-a+b+\sqrt{-(}}\right.\right.}{}$

```
[In] int((e*tan(d*x+c))^(1/2)/(a+cos(d*x+c)*b),x,method=_RETURNVERBOSE)
```

```
[Out] -1/d*((-a^2+b^2)^(1/2)*EllipticPi((-cot(d*x+c)+csc(d*x+c)+1)^(1/2),(a-b)/(a
-b+(-(a-b)*(a+b))^(1/2)),1/2*2^(1/2))-(-a^2+b^2)^(1/2)*EllipticPi((-cot(d*x
+c)+csc(d*x+c)+1)^(1/2),-(a-b)/(-a+b+(-(a-b)*(a+b))^(1/2)),1/2*2^(1/2))-a*E
llipticPi((-cot(d*x+c)+csc(d*x+c)+1)^(1/2),(a-b)/(a-b+(-(a-b)*(a+b))^(1/2))
,1/2*2^(1/2))+EllipticPi((-cot(d*x+c)+csc(d*x+c)+1)^(1/2),(a-b)/(a-b+(-(a-b
)*(a+b))^(1/2)),1/2*2^(1/2))*b-a*EllipticPi((-cot(d*x+c)+csc(d*x+c)+1)^(1/2
),-(a-b)/(-a+b+(-(a-b)*(a+b))^(1/2)),1/2*2^(1/2))+EllipticPi((-cot(d*x+c)+c
sc(d*x+c)+1)^(1/2),-(a-b)/(-a+b+(-(a-b)*(a+b))^(1/2)),1/2*2^(1/2))*b)*(cot(
d*x+c)-csc(d*x+c))^(1/2)*(cot(d*x+c)-csc(d*x+c)+1)^(1/2)*(-cot(d*x+c)+csc(d
*x+c)+1)^(1/2)*(e*tan(d*x+c))^(1/2)*(cot(d*x+c)+csc(d*x+c))*2^(1/2)/(b+(-a^
2+b^2)^(1/2)-a)/(-b+(-a^2+b^2)^(1/2)+a)
```

Fricas [F(-1)]

Timed out.

$$\int \frac{\sqrt{e\tan(c+dx)}}{a+b\cos(c+dx)}\,dx = \text{Timed out}$$

```
[In] integrate((e*tan(d*x+c))^(1/2)/(a+b*cos(d*x+c)),x, algorithm="fricas")
```

```
[Out] Timed out
```

Sympy [F]

$$\int \frac{\sqrt{e\tan(c+dx)}}{a+b\cos(c+dx)}\,dx = \int \frac{\sqrt{e\tan{(c+dx)}}}{a+b\cos{(c+dx)}}\,dx$$

```
[In] integrate((e*tan(d*x+c))**(1/2)/(a+b*cos(d*x+c)),x)
```

```
[Out] Integral(sqrt(e*tan(c + d*x))/(a + b*cos(c + d*x)), x)
```

Maxima [F]

$$\int \frac{\sqrt{e\tan(c+dx)}}{a+b\cos(c+dx)}\,dx = \int \frac{\sqrt{e\tan\left(dx+c\right)}}{b\cos\left(dx+c\right)+a}\,dx$$

```
[In] integrate((e*tan(d*x+c))^(1/2)/(a+b*cos(d*x+c)),x, algorithm="maxima")
[Out] integrate(sqrt(e*tan(d*x + c))/(b*cos(d*x + c) + a), x)
```

Giac [F]

$$\int \frac{\sqrt{e\tan(c+dx)}}{a+b\cos(c+dx)}\,dx = \int \frac{\sqrt{e\tan\left(dx+c\right)}}{b\cos\left(dx+c\right)+a}\,dx$$

```
[In] integrate((e*tan(d*x+c))^(1/2)/(a+b*cos(d*x+c)),x, algorithm="giac")
[Out] integrate(sqrt(e*tan(d*x + c))/(b*cos(d*x + c) + a), x)
```

Mupad [F(-1)]

Timed out.

$$\int \frac{\sqrt{e\tan(c+dx)}}{a+b\cos(c+dx)}\,dx = \int \frac{\sqrt{e\tan\left(c+d\,x\right)}}{a+b\,\cos\left(c+d\,x\right)}\,dx$$

```
[In] int((e*tan(c + d*x))^(1/2)/(a + b*cos(c + d*x)),x)
[Out] int((e*tan(c + d*x))^(1/2)/(a + b*cos(c + d*x)), x)
```

3.22 $\int (a+b\cos(e+fx))^m(g\tan(e+fx))^p\,dx$

Optimal result

Integrand size = 23, antiderivative size = 23

$$\int (a+b\cos(e+fx))^m(g\tan(e+fx))^p\,dx=(g\cot(e+fx))^p(g\tan(e+fx))^p\mathrm{Int}\big((a+b\cos(e+fx))^m(g\cot(e+fx))^{-p},x\big)$$

```
[Out] (g*cot(f*x+e))^p*(g*tan(f*x+e))^p*Unintegrable((a+b*cos(f*x+e))^m/((g*cot(f
*x+e))^p),x)
```

Rubi [N/A]

Not integrable

Time = 0.12 (sec) , antiderivative size = 23, normalized size of antiderivative = 1.00, number of steps used = 0, number of rules used = 0, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.000, Rules used = {}

$$\int (a+b\cos(e+fx))^m(g\tan(e+fx))^p\,dx=\int (a+b\cos(e+fx))^m(g\tan(e+fx))^p\,dx$$

```
[In] Int[(a + b*Cos[e + f*x])^m*(g*Tan[e + f*x])^p,x]
```

```
[Out] (g*Cot[e + f*x])^p*(g*Tan[e + f*x])^p*Defer[Int][(a + b*Cos[e + f*x])^m/(g*
Cot[e + f*x])^p, x]
```

Rubi steps

$$\text{integral}=\left((g\cot(e+fx))^p(g\tan(e+fx))^p\right)\int (a+b\cos(e+fx))^m(g\cot(e+fx))^{-p}\,dx$$

Mathematica [N/A]

Not integrable

Time = 6.95 (sec) , antiderivative size = 25, normalized size of antiderivative = 1.09

$$\int (a+b\cos(e+fx))^m(g\tan(e+fx))^p\,dx=\int (a+b\cos(e+fx))^m(g\tan(e+fx))^p\,dx$$

```
[In] Integrate[(a + b*Cos[e + f*x])^m*(g*Tan[e + f*x])^p,x]
[Out] Integrate[(a + b*Cos[e + f*x])^m*(g*Tan[e + f*x])^p, x]
```

Maple [N/A] (verified)

Not integrable

Time = 1.62 (sec) , antiderivative size = 23, normalized size of antiderivative = 1.00

$$\int \left(a+b\cos\left(fx+e\right)\right)^{m}\left(g\tan\left(fx+e\right)\right)^{p}dx$$

```
[In] int((a+b*cos(f*x+e))^m*(g*tan(f*x+e))^p,x)
[Out] int((a+b*cos(f*x+e))^m*(g*tan(f*x+e))^p,x)
```

Fricas [N/A]

Not integrable

Time = 0.44 (sec) , antiderivative size = 25, normalized size of antiderivative = 1.09

$$\int (a+b\cos(e+fx))^m(g\tan(e+fx))^p\,dx=\int (b\cos\left(fx+e\right)+a)^m(g\tan\left(fx+e\right))^p\,dx$$

```
[In] integrate((a+b*cos(f*x+e))^m*(g*tan(f*x+e))^p,x, algorithm="fricas")
[Out] integral((b*cos(f*x + e) + a)^m*(g*tan(f*x + e))^p, x)
```

Sympy [N/A]

Not integrable

Time = 73.31 (sec) , antiderivative size = 22, normalized size of antiderivative = 0.96

$$\int (a+b\cos(e+fx))^m(g\tan(e+fx))^p\,dx=\int \left(g\tan\left(e+fx\right)\right)^p\left(a+b\cos\left(e+fx\right)\right)^m\,dx$$

```
[In] integrate((a+b*cos(f*x+e))**m*(g*tan(f*x+e))**p,x)
[Out] Integral((g*tan(e + f*x))**p*(a + b*cos(e + f*x))**m, x)
```

Maxima [N/A]

Not integrable

Time = 1.68 (sec) , antiderivative size = 25, normalized size of antiderivative = 1.09

$$\int (a+b\cos(e+fx))^m(g\tan(e+fx))^p\,dx=\int (b\cos(fx+e)+a)^m(g\tan(fx+e))^p\,dx$$

```
[In] integrate((a+b*cos(f*x+e))^m*(g*tan(f*x+e))^p,x, algorithm="maxima")
[Out] integrate((b*cos(f*x + e) + a)^m*(g*tan(f*x + e))^p, x)
```

Giac [N/A]

Not integrable

Time = 1.53 (sec) , antiderivative size = 25, normalized size of antiderivative = 1.09

$$\int (a+b\cos(e+fx))^m(g\tan(e+fx))^p\,dx=\int (b\cos(fx+e)+a)^m(g\tan(fx+e))^p\,dx$$

```
[In] integrate((a+b*cos(f*x+e))^m*(g*tan(f*x+e))^p,x, algorithm="giac")
[Out] integrate((b*cos(f*x + e) + a)^m*(g*tan(f*x + e))^p, x)
```

Mupad [N/A]

Not integrable

Time = 14.78 (sec) , antiderivative size = 25, normalized size of antiderivative = 1.09

$$\int (a+b\cos(e+fx))^m(g\tan(e+fx))^p\,dx=\int (g\tan(e+f\,x))^p\,(a+b\,\cos(e+f\,x))^m\,dx$$

```
[In] int((g*tan(e + f*x))^p*(a + b*cos(e + f*x))^m,x)
[Out] int((g*tan(e + f*x))^p*(a + b*cos(e + f*x))^m, x)
```

CHAPTER 4

APPENDIX

4.1 Listing of Grading functions

The following are the current version of the grading functions used for grading the quality of the antiderivative with reference to the optimal antiderivative included in the test suite.

There is a version for Maple and for Mathematica/Rubi. There is a version for grading Sympy and version for use with Sagemath.

The following are links to the current source code.

The following are the listings of source code of the grading functions.

Mathematica and Rubi grading function

```
(* Original version thanks to Albert Rich emailed on 03/21/2017 *)
(* ::Package:: *)

(* Nasser: April 7,2022. add second output which gives reason for the grade *)
(*                       Small rewrite of logic in main function to make it*)
(*                       match Maple's logic. No change in functionality otherwise*)

(* ::Subsection:: *)
(*GradeAntiderivative[result,optimal]*)


(* ::Text:: *)
(*If result and optimal are mathematical expressions, *)
```

```
(*          GradeAntiderivative[result,optimal] returns*)
(*  "F" if the result fails to integrate an expression that*)
(*      is integrable*)
(*  "C" if result involves higher level functions than necessary*)
(*  "B" if result is more than twice the size of the optimal*)
(*      antiderivative*)
(*  "A" if result can be considered optimal*)


GradeAntiderivative[result_,optimal_] := Module[{expnResult,expnOptimal,leafCountResult,leafCou
    expnResult  = ExpnType[result];
    expnOptimal = ExpnType[optimal];
    leafCountResult = LeafCount[result];
    leafCountOptimal = LeafCount[optimal];

    (*Print["expnResult=",expnResult," expnOptimal=",expnOptimal];*)
    If[expnResult<=expnOptimal,
        If[Not[FreeQ[result,Complex]], (*result contains complex*)
            If[Not[FreeQ[optimal,Complex]], (*optimal contains complex*)
                If[leafCountResult<=2*leafCountOptimal,
                    finalresult={"A",""}
                    ,(*ELSE*)
                    finalresult={"B","Both result and optimal contain complex but leaf count is
                  ]
            ,(*ELSE*)
                finalresult={"C","Result contains complex when optimal does not."}
            ]
        ,(*ELSE*)(*result does not contains complex*)
            If[leafCountResult<=2*leafCountOptimal,
               finalresult={"A",""}
            ,(*ELSE*)
               finalresult={"B","Leaf count is larger than twice the leaf count of optimal. $"<
              ]
        ]
    ,(*ELSE*) (*expnResult>expnOptimal*)
        If[FreeQ[result,Integrate] && FreeQ[result,Int],
            finalresult={"C","Result contains higher order function than in optimal. Order "<>T
            ,
            finalresult={"F","Contains unresolved integral."}
        ]
    ];

    finalresult
]

(* ::Text:: *)
(*The following summarizes the type number assigned an *)
```

```
(*expression based on the functions it involves*)
(*1 = rational function*)
(*2 = algebraic function*)
(*3 = elementary function*)
(*4 = special function*)
(*5 = hyperpergeometric function*)
(*6 = appell function*)
(*7 = rootsum function*)
(*8 = integrate function*)
(*9 = unknown function*)


ExpnType[expn_] :=
  If[AtomQ[expn],
    1,
  If[ListQ[expn],
    Max[Map[ExpnType,expn]],
  If[Head[expn]===Power,
    If[IntegerQ[expn[[2]]],
      ExpnType[expn[[1]]],
    If[Head[expn[[2]]]===Rational,
      If[IntegerQ[expn[[1]]] || Head[expn[[1]]]===Rational,
        1,
      Max[ExpnType[expn[[1]]],2]],
    Max[ExpnType[expn[[1]]],ExpnType[expn[[2]]],3]]],
  If[Head[expn]===Plus || Head[expn]===Times,
    Max[ExpnType[First[expn]],ExpnType[Rest[expn]]],
  If[ElementaryFunctionQ[Head[expn]],
    Max[3,ExpnType[expn[[1]]]],
  If[SpecialFunctionQ[Head[expn]],
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],4]],
  If[HypergeometricFunctionQ[Head[expn]],
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],5]],
  If[AppellFunctionQ[Head[expn]],
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],6]],
  If[Head[expn]===RootSum,
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],7]],
  If[Head[expn]===Integrate || Head[expn]===Int,
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],8]],
  9]]]]]]]]]]]


ElementaryFunctionQ[func_] :=
  MemberQ[{
  Exp,Log,
  Sin,Cos,Tan,Cot,Sec,Csc,
  ArcSin,ArcCos,ArcTan,ArcCot,ArcSec,ArcCsc,
```

```
  Sinh,Cosh,Tanh,Coth,Sech,Csch,
  ArcSinh,ArcCosh,ArcTanh,ArcCoth,ArcSech,ArcCsch
},func]


SpecialFunctionQ[func_] :=
  MemberQ[{
  Erf, Erfc, Erfi,
  FresnelS, FresnelC,
  ExpIntegralE, ExpIntegralEi, LogIntegral,
  SinIntegral, CosIntegral, SinhIntegral, CoshIntegral,
  Gamma, LogGamma, PolyGamma,
  Zeta, PolyLog, ProductLog,
  EllipticF, EllipticE, EllipticPi
},func]


HypergeometricFunctionQ[func_] :=
  MemberQ[{Hypergeometric1F1,Hypergeometric2F1,HypergeometricPFQ},func]


AppellFunctionQ[func_] :=
  MemberQ[{AppellF1},func]
```

Maple grading function

```
# File: GradeAntiderivative.mpl
# Original version thanks to Albert Rich emailed on 03/21/2017

#Nasser 03/22/2017  Use Maple leaf count instead since buildin
#Nasser 03/23/2017  missing 'ln' for ElementaryFunctionQ added
#Nasser 03/24/2017  corrected the check for complex result
#Nasser 10/27/2017  check for leafsize and do not call ExpnType()
#                   if leaf size is "too large". Set at 500,000
#Nasser 12/22/2019  Added debug flag, added 'dilog' to special functions
#                   see problem 156, file Apostol_Problems
#Nasser 4/07/2022   add second output which gives reason for the grade

GradeAntiderivative := proc(result,optimal)
local leaf_count_result,
        leaf_count_optimal,
        ExpnType_result,
        ExpnType_optimal,
        debug:=false;

        leaf_count_result:=leafcount(result);
```

```
        #do NOT call ExpnType() if leaf size is too large. Recursion problem
        if leaf_count_result > 500000 then
            return "B","result has leaf size over 500,000. Avoiding possible recursion issues
        fi;

        leaf_count_optimal := leafcount(optimal);
        ExpnType_result  := ExpnType(result);
        ExpnType_optimal := ExpnType(optimal);

        if debug then
            print("ExpnType_result",ExpnType_result," ExpnType_optimal=",ExpnType_optimal);
        fi;

# If result and optimal are mathematical expressions,
#  GradeAntiderivative[result,optimal] returns
#   "F" if the result fails to integrate an expression that
#       is integrable
#   "C" if result involves higher level functions than necessary
#   "B" if result is more than twice the size of the optimal
#       antiderivative
#   "A" if result can be considered optimal

    #This check below actually is not needed, since I only
    #call this grading only for passed integrals. i.e. I check
    #for "F" before calling this. But no harm of keeping it here.
    #just in case.


    if not type(result,freeof('int')) then
        return "F","Result contains unresolved integral";
    fi;


    if ExpnType_result<=ExpnType_optimal then
        if debug then
             print("ExpnType_result<=ExpnType_optimal");
        fi;
        if is_contains_complex(result) then
            if is_contains_complex(optimal) then
                if debug then
                        print("both result and optimal complex");
                fi;
                if leaf_count_result<=2*leaf_count_optimal then
                    return "A"," ";
                else
                   return "B",cat("Both result and optimal contain complex but leaf count of
                                convert(leaf_count_result,string)," vs. $2 (",
```

```
                                  convert(leaf_count_optimal,string)," ) = ",convert(2*leaf_co
                 end if
             else #result contains complex but optimal is not
                 if debug then
                         print("result contains complex but optimal is not");
                 fi;
                 return "C","Result contains complex when optimal does not.";
             fi;
         else # result do not contain complex
              # this assumes optimal do not as well. No check is needed here.
             if debug then
                    print("result do not contain complex, this assumes optimal do not as well")
             fi;
             if leaf_count_result<=2*leaf_count_optimal then
                 if debug then
                     print("leaf_count_result<=2*leaf_count_optimal");
                 fi;
                 return "A"," ";
             else
                 if debug then
                     print("leaf_count_result>2*leaf_count_optimal");
                 fi;
                 return "B",cat("Leaf count of result is larger than twice the leaf count of opt
                                   convert(leaf_count_result,string),"$ vs. $2(",
                                   convert(leaf_count_optimal,string),")=",convert(2*leaf_count
             fi;
         fi;
     else #ExpnType(result) > ExpnType(optimal)
         if debug then
             print("ExpnType(result) > ExpnType(optimal)");
         fi;
         return "C",cat("Result contains higher order function than in optimal. Order ",
                        convert(ExpnType_result,string)," vs. order ",
                        convert(ExpnType_optimal,string),".");
     fi;

end proc:

#
# is_contains_complex(result)
# takes expressions and returns true if it contains "I" else false
#
#Nasser 032417
is_contains_complex:= proc(expression)
  return (has(expression,I));
end proc:
```

```
# The following summarizes the type number assigned an expression
# based on the functions it involves
# 1 = rational function
# 2 = algebraic function
# 3 = elementary function
# 4 = special function
# 5 = hyperpergeometric function
# 6 = appell function
# 7 = rootsum function
# 8 = integrate function
# 9 = unknown function

ExpnType := proc(expn)
  if type(expn,'atomic') then
    1
  elif type(expn,'list') then
    apply(max,map(ExpnType,expn))
  elif type(expn,'sqrt') then
    if type(op(1,expn),'rational') then
       1
    else
       max(2,ExpnType(op(1,expn)))
    end if
  elif type(expn,'`^`') then
    if type(op(2,expn),'integer') then
      ExpnType(op(1,expn))
    elif type(op(2,expn),'rational') then
      if type(op(1,expn),'rational') then
         1
      else
         max(2,ExpnType(op(1,expn)))
      end if
    else
         max(3,ExpnType(op(1,expn)),ExpnType(op(2,expn)))
    end if
  elif type(expn,'`+`') or type(expn,'`*`') then
    max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
  elif ElementaryFunctionQ(op(0,expn)) then
    max(3,ExpnType(op(1,expn)))
  elif SpecialFunctionQ(op(0,expn)) then
    max(4,apply(max,map(ExpnType,[op(expn)])))
  elif HypergeometricFunctionQ(op(0,expn)) then
    max(5,apply(max,map(ExpnType,[op(expn)])))
  elif AppellFunctionQ(op(0,expn)) then
    max(6,apply(max,map(ExpnType,[op(expn)])))
  elif op(0,expn)='int' then
    max(8,apply(max,map(ExpnType,[op(expn)]))) else
```

```
  9
  end if
end proc:


ElementaryFunctionQ := proc(func)
  member(func,[
        exp,log,ln,
        sin,cos,tan,cot,sec,csc,
        arcsin,arccos,arctan,arccot,arcsec,arccsc,
        sinh,cosh,tanh,coth,sech,csch,
        arcsinh,arccosh,arctanh,arccoth,arcsech,arccsch])
end proc:

SpecialFunctionQ := proc(func)
  member(func,[
        erf,erfc,erfi,
        FresnelS,FresnelC,
        Ei,Ei,Li,Si,Ci,Shi,Chi,
        GAMMA,lnGAMMA,Psi,Zeta,polylog,dilog,LambertW,
        EllipticF,EllipticE,EllipticPi])
end proc:

HypergeometricFunctionQ := proc(func)
  member(func,[Hypergeometric1F1,hypergeom,HypergeometricPFQ])
end proc:

AppellFunctionQ := proc(func)
  member(func,[AppellF1])
end proc:

# u is a sum or product.  rest(u) returns all but the
# first term or factor of u.
rest := proc(u) local v;
  if nops(u)=2 then
     op(2,u)
  else
     apply(op(0,u),op(2..nops(u),u))
  end if
end proc:

#leafcount(u) returns the number of nodes in u.
#Nasser 3/23/17  Replaced by build-in leafCount from package in Maple
leafcount := proc(u)
   MmaTranslator[Mma][LeafCount](u);
end proc:
```

Sympy grading function

```
#Dec 24, 2019. Nasser M. Abbasi:
#             Port of original Maple grading function by
#             Albert Rich to use with Sympy/Python
#Dec 27, 2019  Nasser. Added `RootSum`. See problem 177, Timofeev file
#             added 'exp_polar'
from sympy import *

def leaf_count(expr):
   #sympy do not have leaf count function. This is approximation
   return round(1.7*count_ops(expr))

def is_sqrt(expr):
   if isinstance(expr,Pow):
      if expr.args[1] == Rational(1,2):
         return True
      else:
         return False
   else:
      return False

def is_elementary_function(func):
   return func in [exp,log,ln,sin,cos,tan,cot,sec,csc,
         asin,acos,atan,acot,asec,acsc,sinh,cosh,tanh,coth,sech,csch,
         asinh,acosh,atanh,acoth,asech,acsch
      ]

def is_special_function(func):
   return func in [ erf,erfc,erfi,
          fresnels,fresnelc,Ei,Ei,Li,Si,Ci,Shi,Chi,
          gamma,loggamma,digamma,zeta,polylog,LambertW,
          elliptic_f,elliptic_e,elliptic_pi,exp_polar
       ]

def is_hypergeometric_function(func):
   return func in [hyper]

def is_appell_function(func):
   return func in [appellf1]

def is_atom(expn):
   try:
      if expn.isAtom or isinstance(expn,int) or isinstance(expn,float):
        return True
      else:
        return False
```

```
    except AttributeError as error:
        return False

def expnType(expn):
    debug=False
    if debug:
        print("expn=",expn,"type(expn)=",type(expn))

    if is_atom(expn):
       return 1
    elif isinstance(expn,list):
        return max(map(expnType, expn))    #apply(max,map(ExpnType,expn))
    elif  is_sqrt(expn):
        if isinstance(expn.args[0],Rational): #type(op(1,expn),'rational')
            return 1
        else:
            return max(2,expnType(expn.args[0]))   #max(2,ExpnType(op(1,expn)))
    elif isinstance(expn,Pow):    #type(expn,'`^`')
        if isinstance(expn.args[1],Integer):   #type(op(2,expn),'integer')
            return expnType(expn.args[0])    #ExpnType(op(1,expn))
        elif isinstance(expn.args[1],Rational):   #type(op(2,expn),'rational')
            if isinstance(expn.args[0],Rational): #type(op(1,expn),'rational')
                return 1
            else:
                return max(2,expnType(expn.args[0]))   #max(2,ExpnType(op(1,expn)))
        else:
            return max(3,expnType(expn.args[0]),expnType(expn.args[1]))  #max(3,ExpnType(op(1,expn)),ExpnTy
    elif isinstance(expn,Add) or isinstance(expn,Mul): #type(expn,'`+`') or type(expn,'`*`')
        m1 = expnType(expn.args[0])
        m2 = expnType(list(expn.args[1:]))
        return max(m1,m2)   #max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
    elif is_elementary_function(expn.func):   #ElementaryFunctionQ(op(0,expn))
        return max(3,expnType(expn.args[0]))   #max(3,ExpnType(op(1,expn)))
    elif is_special_function(expn.func): #SpecialFunctionQ(op(0,expn))
        m1 = max(map(expnType, list(expn.args)))
        return max(4,m1)    #max(4,apply(max,map(ExpnType,[op(expn)])))
    elif is_hypergeometric_function(expn.func): #HypergeometricFunctionQ(op(0,expn))
        m1 = max(map(expnType, list(expn.args)))
        return max(5,m1)    #max(5,apply(max,map(ExpnType,[op(expn)])))
    elif is_appell_function(expn.func):
        m1 = max(map(expnType, list(expn.args)))
        return max(6,m1)    #max(5,apply(max,map(ExpnType,[op(expn)])))
    elif isinstance(expn,RootSum):
        m1 = max(map(expnType, list(expn.args))) #Apply[Max,Append[Map[ExpnType,Apply[List,expn]],7]],
        return max(7,m1)
    elif str(expn).find("Integral") != -1:
```

```
        m1 = max(map(expnType, list(expn.args)))
        return max(8,m1)    #max(5,apply(max,map(ExpnType,[op(expn)])))
    else:
        return 9

#main function
def grade_antiderivative(result,optimal):

    #print ("Enter grade_antiderivative for sagemath")
    #print("Enter grade_antiderivative, result=",result," optimal=",optimal)

    leaf_count_result  = leaf_count(result)
    leaf_count_optimal = leaf_count(optimal)

    #print("leaf_count_result=",leaf_count_result)
    #print("leaf_count_optimal=",leaf_count_optimal)

    expnType_result  = expnType(result)
    expnType_optimal = expnType(optimal)

    if str(result).find("Integral") != -1:
        grade = "F"
        grade_annotation =""
    else:
        if expnType_result <= expnType_optimal:
            if result.has(I):
                if optimal.has(I): #both result and optimal complex
                    if leaf_count_result <= 2*leaf_count_optimal:
                        grade = "A"
                        grade_annotation =""
                    else:
                        grade = "B"
                        grade_annotation ="Both result and optimal contain complex but leaf count of result is large
                else: #result contains complex but optimal is not
                    grade = "C"
                    grade_annotation ="Result contains complex when optimal does not."
            else: # result do not contain complex, this assumes optimal do not as well
                if leaf_count_result <= 2*leaf_count_optimal:
                    grade = "A"
                    grade_annotation =""
                else:
                    grade = "B"
                    grade_annotation ="Leaf count of result is larger than twice the leaf count of optimal. "+str(le
        else:
            grade = "C"
            grade_annotation ="Result contains higher order function than in optimal. Order "+str(ExpnType_
```

```
    #print("Before returning. grade=",grade, " grade_annotation=",grade_annotation)

    return grade, grade_annotation
```

SageMath grading function

```
#Dec 24, 2019. Nasser: Ported original Maple grading function by
#          Albert Rich to use with Sagemath. This is used to
#          grade Fricas, Giac and Maxima results.
#Dec 24, 2019. Nasser: Added 'exp_integral_e' and 'sng', 'sin_integral'
#          'arctan2','floor','abs','log_integral'
#June 4, 2022  Made default grade_annotation "none" instead of "" due
#          issue later when reading the file.
#July 14, 2022. Added ellipticF. This is until they fix sagemath, then remove it.

from sage.all import *
from sage.symbolic.operators import add_vararg, mul_vararg

debug=False;

def tree_size(expr):
    r"""
    Return the tree size of this expression.
    """
    #print("Enter tree_size, expr is ",expr)

    if expr not in SR:
        # deal with lists, tuples, vectors
        return 1 + sum(tree_size(a) for a in expr)
    expr = SR(expr)
    x, aa = expr.operator(), expr.operands()
    if x is None:
        return 1
    else:
        return 1 + sum(tree_size(a) for a in aa)

def is_sqrt(expr):
    if expr.operator() == operator.pow:   #isinstance(expr,Pow):
        if expr.operands()[1]==1/2: #expr.args[1] == Rational(1,2):
            if debug: print ("expr is sqrt")
            return True
        else:
            return False
    else:
        return False
```

```
def is_elementary_function(func):
    #debug=False
    m = func.name() in ['exp','log','ln',
            'sin','cos','tan','cot','sec','csc',
            'arcsin','arccos','arctan','arccot','arcsec','arccsc',
            'sinh','cosh','tanh','coth','sech','csch',
            'arcsinh','arccosh','arctanh','arccoth','arcsech','arccsch','sgn',
        'arctan2','floor','abs'
        ]
    if debug:
        if m:
            print ("func ", func , " is elementary_function")
        else:
            print ("func ", func , " is NOT elementary_function")


    return m

def is_special_function(func):
    #debug=False
    if debug:
        print ("type(func)=", type(func))

    m= func.name() in ['erf','erfc','erfi','fresnel_sin','fresnel_cos','Ei',
           'Ei','Li','Si','sin_integral','Ci','cos_integral','Shi','sinh_integral'
           'Chi','cosh_integral','gamma','log_gamma','psi,zeta',
           'polylog','lambert_w','elliptic_f','elliptic_e','ellipticF',
           'elliptic_pi','exp_integral_e','log_integral']

    if debug:
        print ("m=",m)
        if m:
            print ("func ", func ," is special_function")
        else:
            print ("func ", func ," is NOT special_function")


    return m


def is_hypergeometric_function(func):
    return func.name() in ['hypergeometric','hypergeometric_M','hypergeometric_U']

def is_appell_function(func):
    return func.name() in ['hypergeometric']   #[appellf1] can't find this in sagemath
```

```
def is_atom(expn):

  #debug=False
  if debug:
     print ("Enter is_atom, expn=",expn)

  if not hasattr(expn, 'parent'):
    return False


  #thanks to answer at https://ask.sagemath.org/question/49179/what-is-sagemath-equivalent-to-atomic-t
  try:
    if expn.parent() is SR:
      return expn.operator() is None
    if expn.parent() in (ZZ, QQ, AA, QQbar):
      return expn in expn.parent() # Should always return True
    if hasattr(expn.parent(),"base_ring") and hasattr(expn.parent(),"gens"):
      return expn in expn.parent().base_ring() or expn in expn.parent().gens()

    return False

  except AttributeError as error:
    print("Exception,AttributeError in  is_atom")
    print ("cought exception" , type(error).__name__ )
    return False


def expnType(expn):

  if debug:
    print (">>>>>Enter expnType, expn=", expn)
    print (">>>>>is_atom(expn)=", is_atom(expn))

  if is_atom(expn):
    return 1
  elif type(expn)==list:   #isinstance(expn,list):
    return max(map(expnType, expn))   #apply(max,map(ExpnType,expn))
  elif  is_sqrt(expn):
    if  type(expn.operands()[0])==Rational:  #type(isinstance(expn.args[0],Rational):
      return 1
    else:
      return max(2,expnType(expn.operands()[0]))  #max(2,expnType(expn.args[0]))
  elif expn.operator() == operator.pow:   #isinstance(expn,Pow)
    if type(expn.operands()[1])==Integer:   #isinstance(expn.args[1],Integer)
      return expnType(expn.operands()[0])   #expnType(expn.args[0])
    elif type(expn.operands()[1])==Rational:   #isinstance(expn.args[1],Rational)
      if type(expn.operands()[0])==Rational:  #isinstance(expn.args[0],Rational)
```

```
            return 1
         else:
            return max(2,expnType(expn.operands()[0]))  #max(2,expnType(expn.args[0]))
      else:
         return max(3,expnType(expn.operands()[0]),expnType(expn.operands()[1])) #max(3,expnType(expn.
   elif expn.operator() == add_vararg or expn.operator() == mul_vararg: #isinstance(expn,Add) or isinst
      m1 = expnType(expn.operands()[0]) #expnType(expn.args[0])
      m2 = expnType(expn.operands()[1:]) #expnType(list(expn.args[1:]))
      return max(m1,m2)  #max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
   elif is_elementary_function(expn.operator()):  #is_elementary_function(expn.func)
      return max(3,expnType(expn.operands()[0]))
   elif is_special_function(expn.operator()): #is_special_function(expn.func)
      m1 = max(map(expnType, expn.operands()))      #max(map(expnType, list(expn.args)))
      return max(4,m1)   #max(4,m1)
   elif is_hypergeometric_function(expn.operator()): #is_hypergeometric_function(expn.func)
      m1 = max(map(expnType, expn.operands()))      #max(map(expnType, list(expn.args)))
      return max(5,m1)   #max(5,m1)
   elif is_appell_function(expn.operator()):
      m1 = max(map(expnType, expn.operands()))      #max(map(expnType, list(expn.args)))
      return max(6,m1)   #max(6,m1)
   elif str(expn).find("Integral") != -1: #this will never happen, since it
            #is checked before calling the grading function that is passed.
            #but kept it here.
      m1 = max(map(expnType, expn.operands()))      #max(map(expnType, list(expn.args)))
      return max(8,m1)   #max(5,apply(max,map(ExpnType,[op(expn)])))
   else:
      return 9

#main function
def grade_antiderivative(result,optimal):


   if debug:
      print ("Enter grade_antiderivative for sagemath")
      print("Enter grade_antiderivative, result=",result)
      print("Enter grade_antiderivative, optimal=",optimal)
      print("type(anti)=",type(result))
      print("type(optimal)=",type(optimal))

   leaf_count_result  = tree_size(result) #leaf_count(result)
   leaf_count_optimal = tree_size(optimal) #leaf_count(optimal)

   #if debug: print ("leaf_count_result=", leaf_count_result, "leaf_count_optimal=",leaf_count_optimal)


   expnType_result  = expnType(result)
   expnType_optimal = expnType(optimal)
```

```
if debug: print ("expnType_result=", expnType_result, "expnType_optimal=",expnType_optimal)

if expnType_result <= expnType_optimal:
    if result.has(I):
        if optimal.has(I): #both result and optimal complex
            if leaf_count_result <= 2*leaf_count_optimal:
                grade = "A"
                grade_annotation ="none"
            else:
                grade = "B"
                grade_annotation ="Both result and optimal contain complex but leaf count of result is larger th
        else: #result contains complex but optimal is not
            grade = "C"
            grade_annotation ="Result contains complex when optimal does not."
    else: # result do not contain complex, this assumes optimal do not as well
        if leaf_count_result <= 2*leaf_count_optimal:
            grade = "A"
            grade_annotation ="none"
        else:
            grade = "B"
            grade_annotation ="Leaf count of result is larger than twice the leaf count of optimal. "+str(leaf_c
else:
    grade = "C"
    grade_annotation ="Result contains higher order function than in optimal. Order "+str(expnType_resul


print("Before returning. grade=",grade, " grade_annotation=",grade_annotation)

return grade, grade_annotation
```